

国际学术交流英语

数、符号、公式、图形的英文表达

童丽萍 陈治业 编著

东南大学出版社

Expression of Numbers, Signs, Equations and Graphs in English

常言道："完美的公式是数学家的所求"，"图纸是工程师的语言"。同样，在国际学术交流中，如果不能正确地用符合英语习惯的科学语言表达数、符号、公式和图形等，就不可能准确地表达自己的观点，从而很难在国际学术交流舞台上赢得一席之地。

——王光远（我国著名科学家，中国工程院院士）

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·南　京·

内 容 提 要

本书主要介绍了如何用英语正确、科学、规范地表达数、符号、公式和图形。书中给出了大量应用实例，便于科技工作者查阅、参考和引证。本书可作为理工科专业的本科生、研究生、高校教师、各类专业技术人员，以及出国访问进修人员、留学人员和参加国际学术交流人员的参考书和工具书。

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序

近年来，随着我国对外开放政策的不断深入，越来越多的科技工作者加入到国际交流的行列。国际交流工作涉及面广、种类繁多，要求参与者具有较强的外语水平，特别是英语听、说、读、写、译等多项应用技能和对专业知识的熟练掌握和确切的英文表达。但对于我国和其他非英语为母语的国家的学者来说，在对外交流中普遍存在着英语语言方面的困难，这不仅表现在日常英语和专业英语的正确表达，还表现在作为专业基础的诸如数、符号、公式和图形等符合英语习惯的正确表达。

常言道："完美的公式是数学家的所求"，"图纸是工程师的语言"。在国际交流中，如果不能正确用符合英语习惯的科学语言表达类似数、符号、公式和图形等，就不可能准确地表达自己的观点，因而很难在国际学术舞台上赢得一席之地。

为了帮助广大科技工作者系统地掌握国际交流中数、符号、公式和图形的正确英文表达，提高我国学者在学术交流中的语言交际能力和综合竞争能力，本书作者根据在国际交流中的亲身体验和多年从事科技英语翻译工作的经验，参阅了大量国内外相关文献，撰写了此书。

本书比较详细地论述了各种数、符号、公式和图形的口头英语和书面英语表达，就如何用符合英语习惯的科学语言正确表达这些现象进行了深入研究。书中还给出了大量应用实例，以便广大科技工作者查阅、参考和引证。本书具有较高的学术价值和实用价值，是科技工作者从事科学研究和国际交流不可缺少的工具书。虽然目前市面上关于英语学习方面的书很多，但有关国际交流中数、符号、公式和图形的英文表达方面的书籍却不多见，本书的出

版无疑弥补了这方面的不足，故愿为此书作序。

愿本书能为广大科技工作者提供有益的帮助！

王光远

中国工程院院士

2000年8月于哈尔滨

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前　言

这些年来，编者曾有幸多次应邀赴美国、日本、新加坡等国进行学术访问和合作研究，也多次出席国内外举办的国际学术会议。在对外交流中发现，用英语准确、流畅地描述一组公式，娴熟、生动地表达一幅图表并不是一件容易的事情。许多学者在交流中，碰到这类问题时，要么避而不谈，要么用不符合英语习惯的非科学语言表达，结果使对方弄不清交流者想表达和描述些什么，大大影响了交流的质量。令人遗憾的是，这种情况不仅存在于初次进行国际交流的学者身上，那些具有一定语言基础，能熟练进行日常的听、说、读、写的学者，也常常忽略了有关数、符号、公式、图形等的确切的、符合英语习惯的科学表达，致使他们在国际交流的舞台上不能充分发挥才能。目前市面上有关英语学习方面的书甚多，但就如何用英语准确描述或表达数、符号、公式、图形等方面的书却很少。为此，我们编写了《数、符号、公式、图形的英文表达》一书，以求为广大科技工作者提供有益的帮助。

本书共分九章，内容包括数、符号、公式和图形的英文表达。为了便于广大科技工作者查阅、参考和引证，我们在内容上特别强调了符合英语习惯的口头科学语言表达和书面科学语言表达，并列出了大量表达实例。第一章描述了数的英文表达，包括整数、小数和分数以及它们在日常英语及科技英语中的应用；第二章是有关常用数学符号的英文表达，包括一般运算符号、分析符号、函数符号、微分和积分符号、指数和对数符号、三角函数及反三角函数符号、集合符号等，这是学术交流中最基本的符号形式；第三、四、五章是关于简单公式的英语描述，这一部分特别论述了英语的规范表这，给出了许多表达实例，其内容主要包括：初等数学公式的表达、方程和不等式的表达、矩阵和行列式的表达，读者可以参照

这些表达方式进行各种复杂公式的表达；第六、七、八、九章是常见的几何图形的英文表达，包括三角形、四边形、圆等平面图形和立体图形。

本书适用对象为理工科专业的本科生、研究生、高校教师、各类专业技术人员，以及出国访问进修人员、留学人员和参加国际学术交流的人员。

本书在编写过程中受到了我国著名科学家、中国工程院院士王光远教授的鼎力相助，在此表示衷心的感谢。限于作者水平，书中难免有不妥之处，敬请读者批评指正。

编　者

2000 年 8 月

地信网论坛 BBS.3S001.COM

作　者

童丽萍（工程应用学科博士）

多次出国访问讲学、合作研究和出席国际学术会议

陈治业（外国语学院毕业多年的科技英语专业硕士）

具有扎实的语言功底和丰富的翻译实践

读者对象

理工科专业的本科生、研究生、高校教师、各类专业技术人员，以及出国访问进修人员、留学人员和参加国际学术交流的人员

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第一章　数的表达(Numbers)

在数的系统中,包括实数和虚数,如图 1.1 所示。

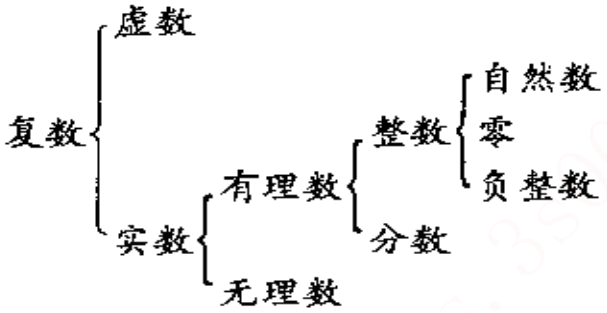


图 1.1　数的系统

The sets of numbers consists of real numbers and pure imaginary numbers, as shown in figure 1.1.

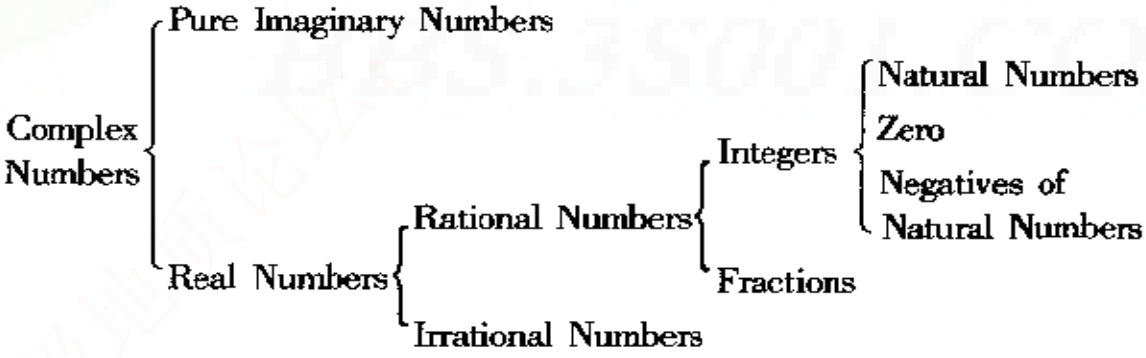


Figure 1.1　Sets of Numbers

数的系统还可用表 1.1 表达.

The sets of numbers can also be expressed in table 1.1.

表 1.1　数的系统（Sets of Numbers）

名　称 Name	符　号 Symbol	实　例 Set
自然数 Natural Numbers	**N**	$\{1,2,3,4,\cdots\}$
整数 Integers	**Z**	$\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$
有理数 Rational Numbers	**Q**	$\frac{16}{3}, 42\frac{1}{2}, 121212\cdots$
无理数 Irrational Numbers	**Q'**	$\sqrt{19}$, $\sqrt{2}$, $\sqrt[5]{43}$, π, $727727772\cdots$
实数 Real Numbers	**R**	$2,0,-14,\frac{43}{48},\sqrt{8},\pi\cdots$
虚数 Pure Imaginary Numbers	**I**	$\sqrt{3}i,\frac{5}{3i},-\frac{2}{17i},65i,i,2i\cdots$
复数 Complex Numbers	**C**	$6,-5,-\frac{3}{2},\frac{14}{31},\sqrt{10}$, $2+3i,+3i,-6i\cdots$

§1.1　整数（Integers）

一、整数的表达（Expression of Integers）

整数包括自然数、零和负整数.

Integers include natural numbers, zero and negative integers.

(一) 整数的分组(Grouping of Integers)

英文中从个位数开始从左到右,每三位数作为一组,用“,”隔开. 对于含有 4 位数或大于 4 位数的整数,分为若干组或节,中间用逗号隔开。

Starting from the units' place and going to the left, separate by commas, the given numeral, when it contains 4 or more figures, into as many groups or periods of 3 figures each as possible.

例 1.1(Example 1.1)　54,562,347,289,642

(二) 整数的读法(How to Read Integers)

与中文相同,英文中整数的读法是从左到右读每一个数.遇到一个逗号,都有一个相应的名称可读,从右起第一个逗号前为“千”,读作“thousand”;第二个逗号前为“百万”,读作“million”;第三个逗号前为“十亿”,读作“billion”;第四个逗号前为“兆”,读作“trillion”.其各个位数的相应读法见图 1.2.

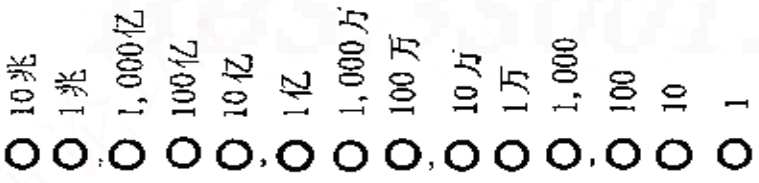


图 1.2　整数的读法

Like in Chinese, when reading integers in English, starting from the left to the right, read each group of figures separately, applying the proper name to the comma as it is reached. The first comma from the right is read "thousand", the second comma "million", the third comma "billion", and the fourth comma "trillion". The corresponding reading is shown in figure 1.2.

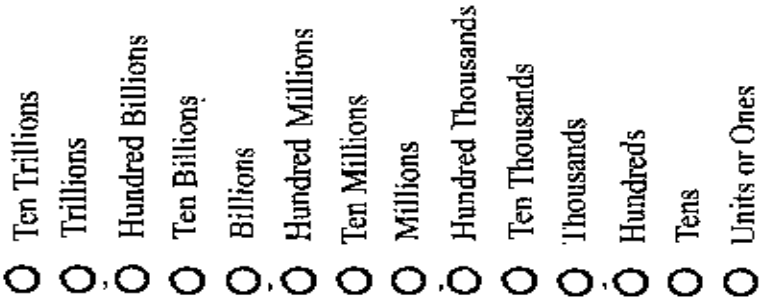


Figure 1.2 Integer Reading

例 1.2(Example 1.2) 35,896,732,546,285

英文读作: thirty-five trillion eight hundred and ninety-six billion seven hundred and thirty-two million five hundred and forty-six thousand two hundred and eighty-five

例 1.3(Example 1.3) 326,784,520,398,216

英文读作: three hundred and twenty-six trillion seven hundred and eighty-four hillion five hundred and twenty million three hundred and ninety-eight thousand two hundred and sixteen

(三) 偶数和奇数(Even and Odd Numbers)

偶数指的是能被 2 整除的数,其个位数为 0、2、4、6、8;不能被 2 整除的数为奇数,其个位数为 1、3、5、7、9.

Even numbers refer to those that can be divided exactly by 2 and whose unit numbers are 0,2,4,6 and 8, and those that cannot be divided by 2 are odd numbers, whose unit numbers are 1,3,5,7 and 9.

例 1.4 下列数中哪些是偶数? 哪些是奇数?

7 6 82 13 600 737 5,428 7,003

答: 6 82 600 5,428 为偶数;

7 13 737 7,003 为奇数.

Example 1.4 Which of the following numbers are even numbers and which are odd ones?

7 6 82 13 600 737 5,428 7,003

Answer 6 82 600 5,428 are even numbers;

7 13 737 7,003 are odd numbers.

(四) 负整数(Negative Integers)

小于零的自然数为负整数,如图 1.3 中数轴原点以左的数.

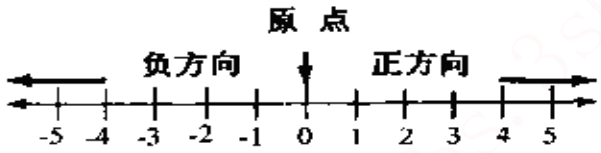


图 1.3 数轴

Those numbers below zero are negative integers. See figure 1.3 in which the numbers on the left of the origin of the number line are negative integers.

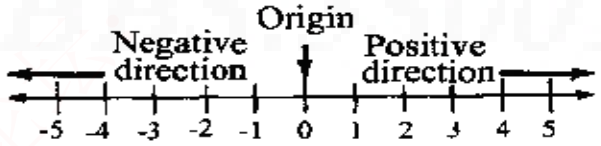


Figure 1.3 Number Line

例 1.5 某地的最高气温为 37℃,最低气温为-12℃.

Example 1.5 The highest temperature in a region is 37℃ and the lowest temperature is -12℃.

(五) 整数的幂 (Powers of Integers)

当遇到一些比较大的数,如太阳的半径大约是 696,000 km,光

的速度大约是3,000,000,000 m/s. 对于这样大的数,读、写都很困难,常常用幂的形式表示,如:

$$696,000 = 696 \times 10^3 \qquad 3,000,000,000 = 3 \times 10^9$$

When meeting some big numbers, such as the radius of the sun which is about 696,000 km, and the velocity of light which is about 3,000,000,000 m/s, it is so hard to read and write such big numbers that they are often expressed as powers. For instance,

$$696,000 = 696 \times 10^3 \qquad 3,000,000,000 = 3 \times 10^9$$

一般地,10的 n 次幂,在1的后面有 n 个0,于是我们就利用10的乘方表示一些大数.例如:

$$1,000,000,000 = 10^9$$

$$\begin{aligned} 453,000,000,000,000 &= 4.53 \times 100,000,000,000,000 \\ &= 4.53 \times 10^{14} \end{aligned}$$

Generally speaking, the nth power of 10 means there are n 0s after 1, so we use the power of 10 to express some big numbers. For instance,

$$1,000,000,000 = 10^9$$

$$\begin{aligned} 453,000,000,000,000 &= 4.53 \times 100,000,000,000,000 \\ &= 4.53 \times 10^{14} \end{aligned}$$

把一个大于10的数记成 $a \times 10^n$ 的形式,其中 a 是整数位只有一位的数,10称为底数,n 为指数,10^n 称作为以10为底的幂,这种表示整数的方法叫做数的幂表示法.

Express a number greater than 10 as the form of $a \times 10^n$, where a is a number with only one digit of an integer, 10 is the base, n is the exponent, and 10^n is the nth power of 10. This expression is called the expression of power.

例1.6 用幂的形式表示下列各数.

1,000,000; 57,000,000

解: $1,000,000 = 10^6$

$57,000,000 = 5.7 \times 10,000,000 = 5.7 \times 10^7$

Example 1.6 Express the following in the form of power.

1,000,000; 57,000,000

Solution $1,000,000 = 10^6$

$57,000,000 = 5.7 \times 10,000,000 = 5.7 \times 10^7$

例 1.7 以下用幂形式表示的数,原来各是什么数?

2×10^6; 5.002×10^7

解: $2 \times 10^6 = 2 \times 1,000,000 = 2,000,000$;

$5.002 \times 10^7 = 5.002 \times 10,000,000 = 50,020,000$

Example 1.7 What is the original number for each of the following expression?

2×10^6; 5.002×10^7

Solution $2 \times 10^6 = 2 \times 1,000,000 = 2,000,000$;

$5.002 \times 10^7 = 5.002 \times 10,000,000 = 50,020,000$

例 1.8 用幂的形式表达下列各数:

(1) 地球离太阳约有一亿五千万千米.

(2) 地球上煤的储量估计为 15 万亿吨以上.

解: (1) $150,000,000 = 1.5 \times 10^8$

答:地球离太阳约有 1.5×10^8 千米.

解: (2) $15,000,000,000,000 = 1.5 \times 10^{13}$

答:地球上煤的储量估计为 1.5×10^{13} 吨以上.

Example 1.8 Use the form of power to express the following:

(1) The distance from the earth to the sun is about 150 million km.

(2) The reserves of coal in the earth is over 15 trillion tons.

Solution (1) $150,000,000 = 1.5 \times 10^8$

Answer The distance from the earth to the sun is about 1.5×10^8 km.

Solution (2) $15,000,000,000,000 = 1.5 \times 10^{13}$

Answer The reserves of coal in the earth is over 1.5×10^{13}tons.

二、基数和序数(Cardinal and Ordinal Numbers)

(一) 基数的读法(How to Read Cardinal Numbers)

在英语中表示数目"多少"的数字叫做基数. 基数的读法如下:

The figure representing the size of a number is called the cardinal number, whose reading is as follows:

阿拉伯数字 (Arabic)	英文读法 (English)	阿拉伯数字 (Arabic)	英文读法 (English)
0	zero /nought	13	thirteen
1	one	14	fourteen
2	two	15	fifteen
3	three	16	sixteen
4	four	17	seventeen
5	five	18	eighteen
6	six	19	nineteen
7	seven	20	twenty
8	eight	21	twenty-one
9	nine	30	thirty
10	ten	100	a/one hundred
11	eleven	101	a/one hundred (and) one
12	twelve		

阿拉伯数字 (Arabic)	英文读法 (English)	阿拉伯数字 (Arabic)	英文读法 (English)
200	two hundred	10^9	one billion
503	five hundred		(英—10^{24})
	and one	10^{12}	a/one trillion
1,000	a/one thousand	10^{15}	one quadrillion
10,000	ten thousand	10^{18}	one quintillion
100,000	a/one hundred		(英—10^{30})
	thousand	10^{21}	one sextillion
1,000,000	a/one million		(英—10^{36})
2,000,000	two million	10^{24}	one septillion
3,004,012	three million,		(英—10^{42})
	four thousand	10^{27}	one octillion
	(and) twelve		(英—10^{48})
12,020,003	twelve million,	10^{30}	one nonillion
	twenty thousand		(英—10^{54})
	(and) three	10^{33}	one decillion
			(英—10^{60})

说明：

1. 21～99 的基数，先说“几十”，再说“几”，中间加连字号.

例如：24—twenty-three　54—fifty-four　79—seventy-nine

2. 101～999 的基数，先说“几百”，再加 and，再加末位数.

例如：156—a/one hundred and fifty-six

426—four hundred and twenty-six

607—six hundred and seven

991—nine hundred and ninety-one

但也有不用 and 的情况，如 850 可以读作 eight hundred fifty.

3. 1,000 以上的数,先从后向前数,每三位数加一",",第一个","前为 thousand,第二个","前为 million, 第三个","前为 billion(美式)或 thousand million(英式),然后一节一节表示.

例如:1,001—one thousand and one

5,000—five thousand

8,436—eight thousand four hundred and thirty-six

284,304—two hundred and eighty-four thousand three hundred and four

20, 654, 693—twenty million six hundred and fifty-four thousand six hundred and ninety-three

850,000,000—eight hundred and fifty million

4. 10 亿以上的数,英式英语和美式英语表达不同.

billion: 1×10^{9}(美式);1×10^{12}(英式),如 1,000,000,000,美式英语读作 a/one billion,英式英语则读作 a/one thousand million.

trillion: 1×10^{12}(美式);1×10^{18}(英式),如 1,000,000,000,000,美式英语读作 a/one trillion,而英式英语读作 a/one billion.

同样,

quadrillion:1×10^{15}(美式); 1×10^{24}(英式)

quintillion: 1×10^{18}(美式); 1×10^{30}(英式)

sextillion: 1×10^{21}(美式); 1×10^{36}(英式)

septillion: 1×10^{24}(美式); 1×10^{42}(英式)

octillion: 1×10^{27}(美式); 1×10^{48}(英式)

nonillion: 1×10^{30}(美式); 1×10^{54}(英式)

decillion: 1×10^{33}(美式); 1×10^{60}(英式)

5. 注意在上述这样的数词中，hundred，thousand，million，billion，trillion 等词一般是单数形式.

例如：two hundred，three million，five trillion，etc.

（二）序数的读法（How to Read Ordinal Numbers）

表示顺序的数字称为序数，如第一、第二、第三等.序数的表达如下表所示：

The figure expressing the sequence of numbers is called the ordinal number，such as first，second and third. Ordinal numbers are expressed as follows：

汉语（Chinese）	缩略形式（Abbreviation）	英语表达（English）
第 1	1st	first
第 2	2nd	second
第 3	3rd	third
第 4	4th	fourth
第 5	5th	fifth
第 6	6th	sixth
第 7	7th	seventh
第 8	8th	eighth
第 9	9th	ninth
第 10	10th	tenth
第 11	11th	eleventh
第 12	12th	twelfth
第 13	13th	thirteenth
第 14	14th	fourteenth
第 15	15th	fifteenth
第 16	16th	sixteenth

第 17	17th	seventeenth
第 18	18th	eighteenth
第 19	19th	nineteenth
第 20	20th	twentieth
第 21	21st	twenty-first
第 22	22nd	twenty-second
第 23	23rd	twenty-third
第 24	24th	twenty-fourth
第 30	30th	thirtieth
第 40	40th	fortieth
第 50	50th	fiftieth
第 60	60th	sixtieth
第 70	70th	seventieth
第 80	80th	eightieth
第 90	90th	ninetieth
第 100	100th	one hundredth
第 101	101st	one hundred and first
第 112	112th	one hundred and twelfth
第 123	123rd	one hundred and twenty-third
第 200	200th	two hundredth
第 230	230th	two hundred and thirtieth
第 365	365th	three hundred and sixty-fifth
第 1000	1,000th	one thousandth
第 2000	2,000th	two thousandth

由上表可以看出,序数词一般以与之相应的基数词加词尾 th

构成，如 fourth(第四)，但也有特别的地方，主要表现在以下几方面：

1. one—first　two—second　three—third　five—fifth　eight—eighth　nine—ninth　twelve—twelfth

2. 以-ty 结尾的词，要先变 y 为 i，再加 -eth，例如：

twenty—twentieth　　forty—fortieth

3. 以 one，two，three，five，eight，nine 收尾的序数词，要照第一条办法变，例如：

thirty-one—thirty-first　　eighty-two—eighty-second

fifty-three—fifty-third　　ninety-five—ninety-fifth

two hundred and fifty-nine—two hundred and fifty-ninth

4. 序数词有时用缩略形式，例如：

first—1st　　second—2nd

fourth—4th　　twenty-third—23rd

two hundred and thirty-fifth—235th

5. 序数词表示顺序时，前面常加定冠词，例如：The First World War，the third lesson. 有时可加一不定冠词来表示“再一”，“又一”等意思，例如：Shall we have to buy a fourth book?（我们还要再买一本书吗？（我们已买了三本书））

(三) 基数和序数的应用

(Application of Cardinal and Ordinal Numbers)

1. 编号(Numbering)

• 第十一课　Lesson 11

读作：lesson eleven 或 the eleventh lesson

- 练习五　Exercise 5

读作：exercise five 或 the fifth exercise

- 第 403 号房间　Room 403

读作：room four oh three

- 第 78 页　page 78

读作：page seventy-eight 或 the seventy-eighth page

- 电话号码 0371-7426329　Telephone Number:0371-7426329

读作：zero three seven one seven four two six three two nine

- 电话号码 3543-1040　Telephone Number: 3543-1040

读作：thirty-five forty-three ten forty

- 97 次列车　Train Number 97

读作：train ninety-seven 或 the ninety-seventh train

- 美国西北航空公司 3644 航班　Flight NW 3644

读作：flight NW three six four four

2. 年、月、日和时刻(year, month, day and time)

★ 年：　年代前用介词 in

- 1976 年　(in) 1976

读作：(in) nineteen seventy-six 或 (in) nineteen hundred and seventy-six

- 1840 年　(in) 1840

读作：(in) eighteen forty 或 (in) eighteen hundred and forty

- 1900 年　(in) 1900

读作：(in) nineteen hundred

- 2000 年　(in) 2000

读作：(in) two thousand

- 公元前 753 年　(in) the year 753 BC

读作：(in) the year seven hundred and fifty-three BC

• 20 世纪 70 年代　1970's 或 1970s
读作:(in) nineteen seventies

★ 月份:　月份开头第一个字母须大写,表示"在某月"时,月份前面用介词 in. 有些月份后附有缩写式.

一月	(in) January Jan.	二月	(in) February Feb.
三月	(in) March Mar.	四月	(in) April Apr.
五月	(in) May	六月	(in) June
七月	(in) July	八月	(in) August Aug.
九月	(in) September Sept.	十月	(in) October Oct.
十一月	(in) November Nov.	十二月	(in) December Dec.

★ 日期:　用序数词(前面须用 the)表示,"在某日"前用介词 on.

十五日	(on) the fifteenth
二十三日	(on) the twenty-third
四日	(on) the fourth
三十一日	(on) the thirty-first

某年某月某日:

• 1945 年 8 月　in August,1945
读作: in August nineteen forty-five

• 1997 年 7 月 1 日　on July 1(st),1997 或 on 1(st) July,1997
读作: on July the first nineteen ninety-seven 或 on the first of July nineteen ninety-seven

★ 时刻:

1) 英语通常用 at 所引导的短语来表示时刻,例如:
我们六点起床。

We get up at six.（或 at six o'clock).

学生们七点二十开始上课。

The pupils begin classes at 7:20.

2）如说“几点整”,用一个数词即可,如说“几点几分”,则用下列方法表示:

(1) 表示“几点过几分”,用介词 past (美式英语可用 after),但分钟须在半小时以内(包括半小时);美式英语中常先说小时,后说分钟. 例如:

7:10　seven ten 或 ten past/after seven

9:15　nine fifteen 或 a quarter past/after nine

11:30　eleven thirty 或 half past eleven

(2) 表示“几点差几分”,用介词 to,也可按顺序读.例如:

10:45,读作 a quarter to eleven 或 ten forty-five

10:50,读作 ten to eleven 或 ten fifty

3）如需说明“上午”、“下午”,可用“a.m.”和“p.m.”. 例如:

- 上午七点　07:00 a.m.

  读作: seven a.m.
- 上午九点十分　09:10 a.m.

  读作: (oh) nine ten a.m.
- 中午　12:00

  读作: midday/noon 或 twelve o'clock
- 下午二点　14:00/2.00 p.m.

  读作: fourteen o'clock 或 two o'clock p.m.
- 晚上十一点二十　23:20/11:20p.m.

  读作: twenty-three twenty 或 eleven twenty p.m.

三、整数的四舍五入（Rounding-off Integers）

在工农业生产及日常生活中，“四舍五入”处理数字是人们习以为常的事情.四舍五入的原则是根据需要保留的数字位数，若第一个数大于5(包括5)，则在前一位数上加1；若第一个数小于5，则去掉该数. 如某地区人口为309,850，若近似到万，我们则可说该地区人口约为31万.

In industry, agriculture and everyday lives, it is common for people to use rounding-off method to deal with numbers. According to the numeral digit needed to preserve, if the first digit is 5 or more, increase the preceding digit by 1, and otherwise, if the first digit is less than 5, remove it. For instance, the population of a district is 309,850, and if rounding off the number to nearest ten thousand, then we can say thet the population of the district is about 310,000.

例 1.9

① 把259四舍五入到十位. 答:260

② 把6,339四舍五入到百位. 答:6,300

③ 把57,734四舍五入到千位. 答:58,000

④ 把483,579四舍五入到万位. 答:480,000

⑤ 把478,695四舍五入到十万位. 答:500,000

⑥ 把47,238,000四舍五入到百万位. 答:47,000,000

Example 1.9

① Round off 259 to nearest ten. The answer is 260.

② Round off 6,339 to nearest hundred. The answer is 6,300.

③ Round off 57,734 to nearest thousand. The answer is 58,000.

④ Round off 483,579 to nearest ten thousand.
The answer is 480,000.

⑤ Round off 478,695 to nearest hundred thousand.

The answer is 500,000.

⑥ Round off 47,238,000 to nearest million.

The answer is 47,000,000.

§1.2 小数和分数(Decimals and Fractions)

一、小数(Decimals)

(一) 小数的写法(How to Write Decimals)

小数由三部分组成,即整数位区、小数位区和小数点.小数点以左为整数位区,从右到左,由个位逐步升高,与整数的排列相同.小数点以右的为小数区,由左向右逐步降低,小数点右边第一位为$(.1=\frac{1}{10})$,第二位为$(.01=\frac{1}{100})$,第三位为$(.001=\frac{1}{1,000})$,第四位为$(.0001=\frac{1}{10,000})$,第五位为$(.00001=\frac{1}{100,000})$,第六位为$(.000001=\frac{1}{1,000,000})$,以此类推.如图 1.4 所示.

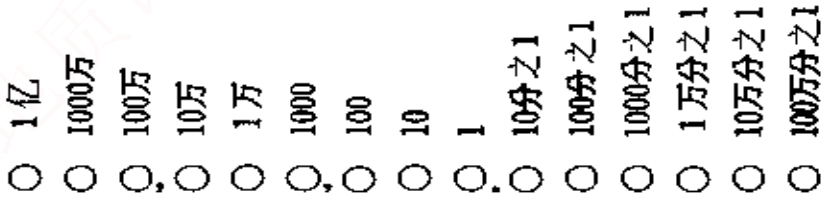


图 1.4 小数的写法

A decimal consists of three parts—whole number, point and decimal fraction. The left side of a point is the area of a whole number and the

right side the decimal area. The first place to the right of the units' place expresses tenth ($.1 = \frac{1}{10}$), the second place hundredth ($.01 = \frac{1}{100}$), the third place thousandth ($.001 = \frac{1}{1,000}$), the fourth place ten-thousandth ($.0001 = \frac{1}{10,000}$), the fifth place hundred-thousandth ($.00001 = \frac{1}{100,000}$), the sixth place millionth ($.000001 = \frac{1}{1,000,000}$), etc., as shown in figure 1.4.

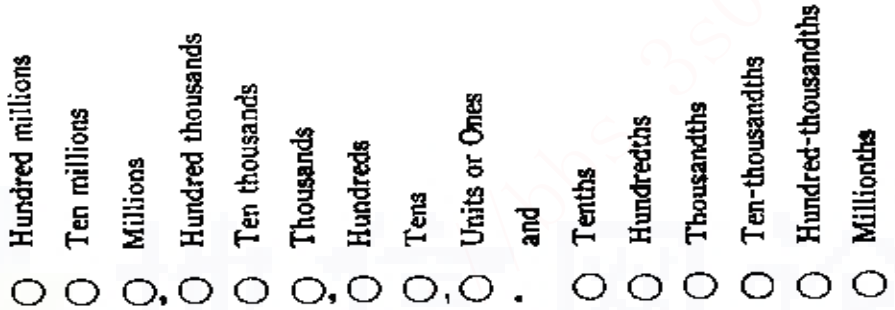


Figure 1.4 Decimal Writing

(二) 小数的表达(How to Express Decimals)

小数部分的表达需分别读出每个数，小数点读作 point，如 3.576，读作"three point five seven six". 若小数的整数位为零，如 0.45，读作"zero point four five"，或"point four five". 若小数是 0.1，0.01，0.001 等，则可读作"one tenth，one hundredth，one thousandth"或"point one，point zero one，point zero zero one，etc."

To express a decimal, we need to read each number and just read point for the decimal point, such as 3.576, which is read "three point five seven six". If the integral part is zero such as 0.45, then the figure is read "zero point four five" or "point four five". If the decimal part is

0.1, 0.01, 0.001, etc., then the figure is read "one tenth, one hundredth, one thousandth" or "point one, point zero one, point zero zero one, etc."

(三) 混合小数(Mixed Decimals)

整数位区不为零的小数称为混合小数,混合小数的读法是整数区按照整数的读法,小数点之后的数字按照小数的读法.

The decimal whose integral part is not zero is called a mixed decimal. When reading a mixed decimal, just read the whole number for the integral place and decimal for the decimal place.

例 1.10(Example 1.10) 329,671,239.953579

英文读作: three hundred and twenty-nine million six hundred and seventy-one thousand two hundred and thirty-nine point nine five three five seven nine

例 1.11(Example 1.11) 2,050.0357

英文读作: two thousand and fifty point zero three five seven

(四) 循环小数和非循环小数 (Recurring and Terminating Decimals)

小数可分为非循环小数和循环小数或重复小数. 如$\frac{2}{5}=0.4$是非循环小数,而$\frac{2}{3}=0.66666\cdots$是循环小数,可写为$0.\overline{6}$,这里的上线"-"表明循环的位数,因此,$\frac{2}{3}=0.\overline{6}$.也可记为$\frac{2}{3}=0.\dot{6}$,这里"·"也表示循环位数.

Decimals can be divided into terminating decimals and recurring or repeating decimals. For instance, $\frac{2}{5}=0.4$ is a terminating decimal and

$0.66666\cdots$ is a recurring or repeating decimal, which can be written as $0.\overline{6}$, where the overbar indicates the digit(s) to be repeated; hence $\frac{2}{3} = 0.\overline{6}$ can also be written as $\frac{2}{3} = 0.\dot{6}$, where the "·" indicates the digit(s) to be repeated as well.

例如下列小数的表达法：

.1	读作：one tenth (point one)
.01	读作：one hundredth (point zero one)
.001	读作：one thousandth (point zero zero one/point two zeros one)
.00001	读作：one millionth (point zero zero zero zero one/point four zeros one)
9.621	读作：nine point six two one
0.347	读作：zero point three four seven
28,367.043	读作：twenty-eight thousand three hundred sixty-seven point zero four three
$0.555 = 0.\dot{5}$	读作：Zero point three fives equals zero point five recurring
$0.\ddot{3}7$	读作：zero point three seven recurring
$.\overline{37}$	读作：point three seven repetend three, seven
$0.3\overline{7}$	读作：point three seven repetend seven
$0.24\ddot{7}3$	读作：zero point two four, seven three recurring/ zero point two four, seven three repetend seven three

(五) 近似数与有效数

(Approximate Numbers and Significant Digits)

一般地,一个近似数,四舍五入到哪一位,就说这个近似数精确到哪一位.这时,从左边第一个不是0的数字起,到精确到的数位止,所有的数字都叫做这个数的有效数字.

Generally speaking, the nearest digit an approximate value rounds off refers to the digit of precision. Here, all the digits from the digit which is not zero on the left to the digit of precision are significant digits.

例 1.12 $3\frac{1}{3}=3.333\cdots$

取 3,就叫做精确到个位;

取 3.3, 就叫做精确到十分位(或精确到 0.1);

取 3.33, 就叫做精确到百分位(或精确到 0.01);

……

3.3 有两个有效数字;

3.33 有三个有效数字.

Example 1.12 $3\frac{1}{3}=3.333\cdots$

Obtaining 3 means correcting to one unit;

Obtaining 3.3 means correcting to one tenth (correct to zero point one);

Obtaining 3.33 means correcting to one hundredth (correct to zero point zero one);

…

3.3 has two significant digits;

3.33 has three significant digits.

例 1.13 下列由四舍五入得到的近似数,各精确到哪一位? 各有哪几个有效数字?

(1) 43.8　　(2) 0.03086　　(3) 2.4 万

解：

(1) 43.8，精确到十分位(即精确到 0.1)，有三个有效数字；

(2) 0.03086，精确到十万分位(即精确到 0.00001)，有四个有效数字；

(3) 2.4 万，精确到千位，有两个有效数字.

Example 1.13　In the following approximate numbers by rounding-off, to which digit is each of them correct? How many significant digits does each have?

Solution

(1) 43.8 is correct to one tenth (correct to zero point one) and there are three significant digits;

(2) 0.03086 is correct to one hundred-thousandths (correct to zero point four zeros one) and there are four significant digits;

(3) 2.4 ten thousand is correct to one thousandth and there are two significant digits.

(六) 小数的幂表达 (Power Expression of Decimals)

当遇到较小的数时，如，-0.00016，0.0000000125 等，这样小的数，读、写同样很不方便，常常需要用幂的形式表达. 如

将 $\frac{1}{10^2}$ 记作 10^{-2}；同理将 $\frac{1}{10^n}$ 记作 10^{-n}

For very small numbers, such as -0.00016, 0.0000000125, etc., they are still not easy to read and write, and are often expressed in the form of powers. For instance,

$\frac{1}{10^2}$ is written as 10^{-2}, and similarly, $\frac{1}{10^n}$ is written as 10^{-n}

例 1.14(Example 1.14)　$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$

英文读作：Zero point zero one equals one over one hundred equals one over ten squared equals ten to the minus second power.

例 1.15(Example 1.15)　$0.00001 = \frac{1}{100000} = \frac{1}{10^5} = 10^{-5}$

英文读作：Zero point four zeros one equals one over one hundred thousand equals one over ten to the fifth power equals ten to the minus fifth power.

例 1.16(Example 1.16)　$0.000007 = 7 \times 10^{-6}$

英文读作：Zero point five zeros seven equals seven times ten to the minus sixth power

例 1.17(Example 1.17)

用幂的表示法表达下列各数.

Express the following as powers.

(1) 0.004　　(2) −0.0000053　　(3) 0.0000000391

解(Solution)

(1) $0.004 = 4 \times 0.001 = 4 \times 10^{-3}$

英文读作：Zero point zero zero four equals four times zero point zero zero one equals four times ten to the minus third power.

解(Solution)

(2) $-0.0000053 = -5.3 \times 0.000001 = -5.3 \times 10^{-6}$

英文读作：Minus zero point five zeros five three equals minus five point three times zero point five zeros one equals minus five point three times ten to the minus sixth power.

解(Solution)

(3) $0.0000000391 = 3.91 \times 0.00000001 = 3.91 \times 10^{-8}$

英文读作：Zero point seven zeros three nine one equals three point nine

one times zero point seven zeros one equals three point nine one times ten to the minus eighth power.

例 1.18　一种细菌的半径是 4×10^{-5}m，用小数将其表示出来.（单位仍用 m）

解：$4\times10^{-5}=4.0\times0.00001=0.00004$（m）

答：这种细菌的半径为 0.00004 m.

Example 1.18　The radius of a bacterium is 4×10^{-5}m. Express the number in a decimal. (The unit is meter, too)

Solution　$4\times10^{-5}=4.0\times0.00001=0.00004$（m）

英文读作：Four times ten to the minus fifth power equals four point zero times zero point four zeros one equals zero point four zeros four meters.

Answer　The radius of the bacterium is 0.00004 meters.

二、分数(Fractions)

(一) 分数的表达(How to Express Fractions)

分数由分子、分母和分数线所组成. 在英语中，一般用基数词代表分子，序数词代表分母，除了分子是"1"的情况外，序数词都要用复数.

A fraction consists of a numerator, a denominator and a fraction stroke. In English, we usually use a cardinal to express the numerator and an ordinal to express the denominator. Except that the numerator is 1, all the cardinals must be plural.

如(For example):

分　数	英文表达	分　数	英文表达
$\frac{1}{4}$	one-fourth	$\frac{2}{3}$	two-thirds
$\frac{7}{9}$	seven-ninths	$\frac{5}{12}$	five-twelfths
$2\frac{3}{5}$	two and three-fifths	$30\frac{7}{8}$	thirty and seven-eighths

另外,还有下面一些表示法:

Otherwise, there are still other expressions:

分　数	英文表达	分　数	英文表达
$\frac{1}{2}$	a (one) half	$\frac{1}{4}$	a (one) quarter
$\frac{3}{4}$	three-quarters	$2\frac{1}{4}$	two and a quarter
$1\frac{1}{2}$	one and a half	$5\frac{1}{3}$	five and a third

比较复杂的分数读法如下:

More complicated fractions are read as follows:

$\frac{2}{3}$ two over three (two-thirds 或 two divided by three)

$\frac{22}{9}$ twenty-two over nine (twenty-two divided by nine)

$\frac{33}{89}$ thirty-three over eighty-nine (thirty-three divided by eighty-nine)

$3\frac{4}{7}$ three and four over seven (three and four-sevenths)

$5\frac{5}{8}$ five and five-eighths (five and five over eight 或 five and five divided by eight)

表达实例(Examples for expression)

$\frac{1}{2}$ ····· one half

$\frac{1}{3}$ ····· one third

$\frac{1}{4}$ ····· one fourth (a/one quarter)

$\frac{1}{5}$ ····· one fifth

$\frac{1}{10}$ ····· one tenth

$\frac{1}{20}$ ····· one twentieth

$\frac{1}{21}$ ····· one twenty-first (one over twenty-one)

$\frac{2}{21}$ ····· two twenty-firsts (two over twenty-one)

$\frac{5}{22}$ ····· five twenty-seconds (five over twenty-two)

$\frac{7}{100}$ ····· seven hundredths (seven one-hundredths 或 seven over a/one hundred)

$\frac{3}{4}$ ····· three quarters (three fourths)

$\frac{2}{5}$ ····· two fifths

$\frac{123}{567}$ ····· one hundred and twenty-three five hundred and sixty-sevenths (one hundred and twenty-three over five hundred and sixty-seven)

$3\frac{1}{3}$ ····· three and one-third

$5\frac{7}{8}$ ····· five and seven-eighths

(二) 分数与小数的转换(Conversion of Fractions and Decimals)

分数与小数可以相互转换,如$\frac{3}{8}=0.375$, $-0.6=-\frac{3}{5}$等.

A fraction and a decimal can be converted into each other, such as $\frac{3}{8}=0.375$ and $-0.6=-\frac{3}{5}$.

例 1.19(Example 1.19)

将下列每个分数变为小数.

Rewrite each fraction as a decimal.

(1) $\frac{6}{10}$ (2) $\frac{8}{25}$ (3) $-\frac{17}{6}$ (4) $\frac{3}{7}$

解 (Solution): (1) $\frac{6}{10}=0.6$

英文读作:Six divided by ten equals zero point six.

解 (Solution): (2) $\frac{8}{25}=0.32$

英文读作:Eight over twenty-five equals zero point three two.

解 (Solution): (3) $-\frac{17}{6}=-2.83333\cdots=-2.8\dot{3}$

英文读作:Minus seventeen divided by six equals minus two point eight four threes, etc. equals minus two point eight, three repetend three.

解 (Solution): (4) $\frac{3}{7}=0.428571428571\cdots=0.\overline{428571}$

英文读作:Three divided by seven equals zero point four two eight five seven one four two eight five seven one, etc. equals zero point four two eight five seven one repetend four two eight five seven one.

例 1.20(Example 1.20)

将下列每个小数变为分数.

Convert each decimal below into a fraction.

(1) 0.7　　(2) −0.63　　(3) 1.075

解(Solution): (1) $0.7 = \frac{7}{10}$

英文读作: Zero point seven equals seven over ten.

解(Solution): (2) $-0.63 = -\frac{63}{100}$

英文读作: Minus zero point six three equals minus sixty-three over one hundred.

解(Solution): (3) $1.075 = \frac{1075}{1000} = \frac{43}{40}$

英文读作: One point zero seven five equals one thousand and seventy-five over one thousand equals forty-three over forty.

(三) 百分数和千分数(Percent and Permille)

分数和小数常常可以用百分数或千分数表示,如 3% 表示$\frac{3}{100}$或 0.03;5‰表示$\frac{5}{1000}$或 0.005.

Fractions and decimals are often expressed in percent or permille. For instance, 3% means $\frac{3}{100}$ or 0.03; 5‰ means $\frac{5}{1000}$ or 0.005.

例 1.21(Example 1.21)

将下列百分数变为小数.

Rewrite each percent below as a decimal.

(1) 5.7%　　(2) 0.003%

解(Solution): (1) $5.7\% = \frac{5.7}{100} = 0.057$

英文读作：Five point seven percent equals five point seven over one hundred equals zero point zero five seven.

解(Solution)：　(2) $0.003\% = \frac{0.003}{100} = 0.00003$

英文读作：Zero point zero zero three percent equals zero point zero zero three over one hundred equals zero point four zeros three.

例 1.22(Example 1.22)

将下列分数变为百分数.

Rewrite each fraction below as a percent.

(1) $\frac{4}{5}$　　(2) $\frac{1}{3}$

解(Solution)：　(1) $\frac{4}{5} = 0.8 = 0.8 \times 100\% = 80\%$

英文读作：Four over five equals zero point eight equals zero point eight multiplied by one hundred percent equals eighty percent.

解(Solution)：　(2) $\frac{1}{3} = 0.\dot{3} = 0.\dot{3} \times 100\% = 33.\dot{3}\%$

英文读作：One third equals zero point three repetend three equals zero point three repetend three multiplied by one hundred percent equals thirty-three point three repetend three percent.

例 1.23(Example 1.23)

将下列小数和分数变为千分数.

Rewrite the following decimals or fractions as a permille.

(1) 0.0008　　(2) $\frac{1}{4}$

解(Solution)：　(1) $0.0008 = \frac{8}{10000} = 0.8‰$

英文读作：Zero point zero zero zero eight equals eight over ten thousand equals zero point eight permille.

解(Solution)：　(2) $\frac{1}{4}=0.25=0.25\times1000‰=250‰$

英文读作：One fourth equals zero point two five equals zero point two five times one thousand permille equals two hundred and fifty permille.

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第二章　符号的表达(Signs)

§2.1　一般数学符号(Basic Mathematical Signs)

一、基本运算符号(General Signs of Operation)

(一) +　加号;正号(Plus; Positive)

例 2.1(Example 2.1)　(1) $2+3$　(2) $a+5$　(3) $+7$

英文读作:(1) two plus three

(2) a plus five

(3) positive/plus seven

(二) −　减号;负号(Minus;Negative)

例 2.2(Example 2.2)　(1) $8-4$　(2) $10-b$　(3) $-3\frac{3}{5}$

英文读作:(1) eight minus four

(2) ten minus b

(3) negative/minus three and three-fifths

(三) ±;∓　正负;负正(Plus or Minus; Minus or Plus)

例 2.3(Example 2.3)　(1) $x=\pm 3$　(2) $y=\mp 5$

英文读作:(1) x equals plus or mimus three

(2) y equals minus or plus five

(四) ×； · 乘号(multiplied by; times)

例 2.4(Example 2.4) (1) 5×6 (2) $x\cdot y$

英文读作：(1) five times six 或 five multiplied by six

(2) x times y 或 x multiplied by y

(五) —； /； ÷ 除号(over; is to; divided by)

例 2.5(Example 2.5) (1) $\frac{3}{4}$ (2) $9\div4$ (3) s/t

英文读作：(1) three over four 或 three quarters

(2) nine divided by four

(3) s divided by t

(六)： 比例(Proportion)

例 2.6(Example 2.6) (1) 1:2 (2) $a:b$

英文读作：(1) the ratio of one to two

(2) a is to b

(七) % 百分比(Percent; Per cent)

例 2.7(Example 2.7) (1) 2% (2) 0.3%

英文读作：(1) two per cent 或 two percent

(2) zero point three percent

(八) ‰ 千分比(Permille; Per mille)

例 2.8(Example 2.8) (1) 5‰ (2) 3.8‰

英文读作：(1) five per mille

(2) three point eight per mille

二、括号(Signs of Grouping)

(一)(　)　圆括号(Round Brackets; Parentheses)

例 2.9(Example 2.9)　$\left(6+5\frac{3}{4}-4.66\times 3\right)\div 3\frac{1}{7}$

英文读作：open parenthesis six plus five and three-fourths minus four point six six multiplied by three close parenthesis divided by three and one seventh

例 2.10(Example 2.10)　$(a+b)c$

英文读作：the quantity a plus b times c

例 2.11(Example 2.11)　$a^2-b^2=(a+b)(a-b)$

英文读作：a squared minus b squared equals the quantity a plus b times the quantity a minus b .

或：a squared minus b squared equals open parenthesis a plus b close parenthesis times open parenthesis a minus b close parenthesis.

例 2.12(Example 2.12)　$(a-b)^2=a^2-2ab+b^2$

英文读作：The quantity a minus b squared equals a squared minus two ab plus b squared.

例 2.13(Example 2.13)

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\cdots+b^n$$

英文读作：The quantity a plus b to the nth power equals a to the nth power plus n times the quantity a to the n minus one power times b plus n times the quantity n minus one divided by two

times the quantity a to the n minus two power times b squared down to plus b to the nth power.

例 2.14(Example 2.14)　$k = 81\left(\frac{2}{3}\right)^8$

英文读作：k equals eighty-one times two-thirds to the eighth power.

例 2.15(Example 2.15)　$\frac{a}{b(c+d)}$

英文读作：a divided by the whole quantity b times c plus d

例 2.16(Example 2.16)　$(\frac{a}{b}c + d)$

英文读作：open parenthesis a over b times c plus d close parenthesis

例 2.17(Example 2.17)　$\frac{1}{x^2(x^2-4)^3}$

英文读作：one over the quantity x squared times the quantity x squared minus four to the third power

例 2.18(Example 2.18)　$\frac{x^3+1}{(x^2+1)^{\frac{2}{3}}}$

英文读作：x to the third power plus one over/divided by the quantity x squared plus one to the two-thirds power

例 2.19(Example 2.19)　$\lim\limits_{n\to\infty}\frac{1}{n^2}(1+2+\cdots+n)$

英文读作：the limit as n approaches infinity of the quantity one over n squared times the sum of n positive integers

或：　the limit as n approaches infinity of the quantity one over n squared times one plus two plus up to plus n

(二)〔　〕 方括号(Square Brackets)

例 2.20(Example 2.20)　$15x^2 - [-4x^2 + 6(x - x^2) - 3x]$

英文读作：fifteen times x squared minus open bracket minus four times x squared plus six open parenthesis x minus x squared close parenthesis minus three times x close bracket

(三)｛　｝ 花括号(Braces)

例 2.21(Example 2.21)　$\frac{1}{2}\{a[b + (c - d)]\}$

英文读作：one-half open brace, a open bracket b plus open parenthesis c minus d close parenthesis close bracket close brace

三、等号和不等号(Signs of Equality and Inequality)

(一) = 等于(be equal to; equals)

例 2.22(Example 2.22)　$k = 81\left(\frac{2}{3}\right)^8$

英文读作：k equals eighty-one times two-thirds to the eighth power.

例 2.23(Example 2.23)　$(a - b)(a + b) = a^2 - b^2$

英文读作：The product of the sum and difference of two terms a and b equals the difference of their squares.

(二) ≡ 恒等于(be identical with/to; be equivalent to)

例 2.24(Example 2.24)　$c \equiv 5$

英文读作：c is identical with five/ c is equivalent to five/ c is identical to five.

(三) $\neq$ 不等于(be not equal to)

例 2.25(Example 2.25) $c + d \cdot a \neq 0$

英文读作：c plus d times a is not equal to zero.

(四) $\not\equiv$ 不恒等于(be not identical with)

例 2.26(Example 2.26) $x \not\equiv 0$

英文读作：x is not identical with/to zero.

(五) $\propto$ 正比于(be in direct proportion to)

例 2.27(Example 2.27) $G = mg$，这里 $G \propto m$

英文读作：G equals m times g , and G is in direct proportion to m .

(六) $\approx$ 近似等于(be approximately equal to)

例 2.28(Example 2.28) $t = \frac{3}{7} \approx 0.429$

英文读作：t equals three divided by seven is approximately equal to zero point four two nine.

(七) $\triangleq$ 对应于(be corresponding to)

例 2.29(Example 2.29) $\frac{2\pi}{4} \triangleq 90^\circ$

英文读作：Two π over four is corresponding to ninety degrees.

(八) $<$； $>$ 小于；大于(be less than; be greater than)

例 2.30(Example 2.30) (1) $a < b$ (2) $x > 0$

英文读作：(1) a is less than b ;

(2) x is greater than zero.

(九) $\leq$; $\geq$ 小于或等于;大于或等于(be less than or equal to; be greater than or equal to)

例 2.31(Example 2.31)

$x \leq 6$; $x \geq -6$ 可写成: $-6 \leq x \leq 6$

英文读作: x is less than or equal to six; x is greater than or equal to minus six, which can be written as x is greater than or equal to minus six and less than or equal to six.

(十) $\ll$; $\gg$ 远远小于; 远远大于(be far less than; be far greater than)

例 2.32(Example 2.32) (1) $c \ll 1{,}000$ (2) $c \gg -3$

英文读作: (1) c is far less than one thousand;

(2) c is far greater than minus three.

§2.2 初等代数符号(Signs of Elementary Algebra)

一、分析符号(Signs of Analysis)

(一) | | 绝对值(Absolute Value)

例 2.33(Example 2.33)

如果(If) $|a| = |b|$; $|a| = \pm a$; $|b| = \pm b$,则(then)

$\pm a = \pm b$

反之,如果 $\pm a = \pm b$,则(Conversely, if $\pm$ a $=\pm$ b, then)

$|a| = |b|$

英文读作: If the absolute value of a equals that of b, the absolute value of a equals plus or minus a and the absolute value of

b equals plus or minus b , then plus or minus a equals plus or minus b . Reversing this reasoning, if plus or minus a equals plus or minus b , then the absolute value of a equals that of b .

例 2.34(Example 2.34)　$|2x + 1| = 7$

英文读作：The absolute value of 2 times x plus one equals seven.

(二) $n!$ (Factorial n)

n 阶乘，　! 为阶乘记号.

Factorial n , where ! is the sign of factorial.

例 2.35(Example 2.35)　$5! = 5 \cdot 4! = 5 \cdot 4 \cdot 3! = 5 \cdot 4 \cdot 3 \cdot 2!$

英文读作：Five factorial equals five times four factorial equals five times four times three factorial equals five times four times three times two factorial.

例 2.36(Example 2.36)

$n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$

英文读作：n factorial equals n times the quantity n minus one factorial equals n times the quantity n minus one times the quantity n minus two factorial equals n times the quantity n minus one times the quantity n minus two times the quantity n minus three factorial.

(三) $\sum$　求和(the Sum of the Terms Indicated; Summation of)

例 2.37　求　$\sum_{n=1}^{5} S_n$；其中，$S_n = \frac{n}{n+1}$.

解：$\sum_{n=1}^{5} S_n = S_1 + S_2 + S_3 + S_4 + S_5$

用数字代替每个 S_n

得：$\sum_{n=1}^{5} S_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$

$$= \frac{30 + 40 + 45 + 48 + 50}{60}$$

$$= \frac{213}{60} = \frac{71}{20}$$

Example 2.37 Find $\sum_{n=1}^{5} S_n$, where $S_n = \frac{n}{n+1}$.

Solution

$$\sum_{n=1}^{5} S_n = S_1 + S_2 + S_3 + S_4 + S_5$$

英文读作：Capital sigma S_n from n equals one to n equals five i.e. add the terms S_n for consecutive values of n, starting with n equals one up to and including n equals five equals S_1 plus S_2 plus S_3 plus S_4 plus S_5.

Now replace each S_n by its numerical value.

Obtain

$$\sum_{n=1}^{5} S_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

$$= \frac{30 + 40 + 45 + 48 + 50}{60}$$

$$= \frac{213}{60} = \frac{71}{20}$$

英文读作：Capital sigma S_n from n equals one to n equals five equals a half plus two thirds plus three quarters plus four fifths plus five sixths equals thirty plus forty plus forty-five plus forty-eight plus fifty over sixty equals two hundred and thirteen over sixty equals seventy-one over twenty.

(四) ∏ 连乘积(the Product of the Terms Indicated)

例 2.38(Example 2.38)

(1) $\prod_{n=1}^{\infty} a_n$ (2) $\prod_{k=1}^{n} a_k$

英文读作:(1) product of all a_n from n equals one to infinity

(2) product of all a_k, *from* k equals one to n

(五) max 最大值(maximum value);
min 最小值(minimum value)

例 2.39(Example 2.39) (1) max $f(x)$ (2) min $f(x)$

英文读作:(1) maximum value of $f(x)$

(2) minimum value of $f(x)$

例 2.40(Example 2.40)

(1) max $(a_1, \cdots, a_n)$ (2) min $(a_1, \cdots, a_n)$

英文读作:(1) maximum value of the series a sub one to a sub n

(2) minimum value of the series a sub one to a sub n

二、乘方和开方(Powers and Roots)

(一) 乘方(Powers)

乘方记作 x^n,表示 x 的 n 次方.

Power is written as x^n, expressing the nth power of x or x to the nth power.

例 2.41(Example 2.41)

(1) $3^2 = 9$ (2) $3^3 = 27$ (3) $2^4 = 16$

英文读作:(1) Three squared equals nine;

(2) Three cubed equals twenty-seven;

(3) Two to the fourth power equals sixteen.

例 2.42(Example 2.42)

(1) x^2 (2) x^3 (3) x^4 (4) x^n

英文读作：(1) x squared (2) x cubed

(3) x to the fourth power (4) x to the nth power

例 2.43(Example 2.43)

(1) $K = ar^{n-1}$ (2) $x^5x^6 = x^{5+6} = x^{11}$

英文读作：(1) K equals a times r to the n minus one power;

(2) x to the fifth power times x to the sixth power equals x to the five plus six power equals x to the eleventh power.

(二) 开方(Root-extracting)

开方记作 $\sqrt[n]{x}$ 或 $x^{\frac{1}{n}}$，表示 x 的 n 次根或 x 的 $\frac{1}{n}$ 次方，当 $n = 2$ 时，表示开平方根，2 省略.

The extraction of root is written as $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$, which expresses nth root of x or x to the quantity one over n power and when $n = 2$, it expresses extraction of square root, where 2 is left out.

例 2.44(Example 2.44) (1) $\sqrt[n]{a} = b$ (2) $\sqrt{16} = 4$

英文读作：(1) The principal nth root of a is b;

(2) The principal square root of 16 is 4.

例 2.45 如果 $\sqrt{a} = b$，则 $b^2 = a$；即 $(\sqrt{a})^2 = a$. 同理，如果 $\sqrt[3]{a} = b$，则 $b^3 = a$，或 $(\sqrt[3]{a})^3 = a$. 总之，如果 n 为任意正整数，则

$$(\sqrt[n]{a})^n = a$$

Example 2.45 If $\sqrt{a} = b$, then $b^2 = a$; that is, $(\sqrt{a})^2 = a$. Similarly, if $\sqrt[3]{a} = b$, then $b^3 = a$, or $(\sqrt[3]{a})^3 = a$. In general, if n is any positive integer, then

$$(\sqrt[n]{a})^n = a$$

英文读作：The quantity principal nth root of a to the nth power is a .

三、虚数和 2 个常用无理数

（Imaginary Numbers and Two Common Irrational Numbers）

（一）虚数 i（Imaginary Number i）

在复数中，i 是一种虚数记号，其定义为

$$i = \sqrt{-1} \quad 和 \quad i^2 = -1$$

用 i，一个负实数的平方根定义为

$$\sqrt{-x} = \sqrt{-1}\sqrt{x} = i\sqrt{x} \qquad 当\ x > 0\ 时$$

In complex numbers, i is used as a symbol of an imaginary number and is defined as follows:

$$i = \sqrt{-1} \quad \text{and} \quad i^2 = -1$$

Using i, the square root of a negative real number is defined by:

$$\sqrt{-x} = \sqrt{-1}\sqrt{x} = i\sqrt{x} \qquad \text{for } x > 0$$

例 2.46 简化 (1) $\sqrt{-16} + \sqrt{-25}$; (2) $\sqrt{-16} \cdot \sqrt{-25}$

解：$\because \sqrt{-16} = \sqrt{-1} \cdot \sqrt{16} = i \cdot 4 = 4i$

$$\sqrt{-25} = \sqrt{-1} \cdot \sqrt{25} = i \cdot 5 = 5i$$

$\therefore$ (1) $\sqrt{-16} + \sqrt{-25} = 4i + 5i = 9i$

(2) $\sqrt{-16} \cdot \sqrt{-25} = 4i \cdot 5i = 20i^2 = 20(-1) = -20$

Example 2.46

Simplify (1) $\sqrt{-16}+\sqrt{-25}$; (2) $\sqrt{-16}\cdot\sqrt{-25}$

Solution

Since $\sqrt{-16}=\sqrt{-1}\cdot\sqrt{16}=\mathrm{i}\cdot 4=4\mathrm{i}$

$\sqrt{-25}=\sqrt{-1}\cdot\sqrt{25}=\mathrm{i}\cdot 5=5\mathrm{i}$,

(1) $\sqrt{-16}+\sqrt{-25}=4\mathrm{i}+5\mathrm{i}=9\mathrm{i}$.

英文读作: The square root of minus sixteen plus the square root of minus twenty-five equals four times i plus five times i equals nine times i

(2) $\sqrt{-16}\cdot\sqrt{-25}=4\mathrm{i}5\mathrm{i}=20\mathrm{i}^2=20(-1)=-20$

英文读作: The square root of minus sixteen times the square root of minus twenty-five equals four i times five i equals twenty i squared equals twenty times minus one equals minus twenty.

(二) 无理数 π(pi) (Irrational Number π)

π 为圆周率,其值约为 3.14159.

π is the ratio of the circumference of a circle to its diameter, approx. 3.14159.

例 2.47 圆周的面积为 $A=\pi R^2$ (R 为圆周的半径).

Example 2.47 The area of the circumference of a circle $A=\pi R^2$ (R is the radius of the circumference).

例 2.48 圆的周长为 $C=\pi D$ (D 为圆的直径).

Example 2.48 The perimeter of a circle $C=\pi D$ (D is the diameter of the circle).

(三) 无理数 e (Irrational Number e)

e 称为自然对数底,约等于 2.71828.

e is the base of a natural logarithm, approx. 2.71828.

例 2.49(Example 2.49)

(1) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

(2) $\ln x = \log_e x$ for every $x > 0$

英文读作:(1) Limit as n approaches infinity of the quantity of one plus one over n to the nth power equals e;

(2) ell-en of x equals the natural logarithm of x, for every x is greater than zero.

§2.3 简单函数与区间(Functions and Intervals)

一、函数(Functions)

(一) f 或 F

F 或 f 是函数的记号,二次函数通常表示为

$$f(x) = ax^2 + bx + c,\ a \neq 0$$

F or f is the sign of function and the quadratic function is usually defined by

$$f(x) = ax^2 + bx + c,\ a \neq 0$$

英文读作: The function of x equals a times the square of x plus b times x plus c, where a is not equal to zero.

例 2.50(Example 2.50)

$$f(x) = 9x^2 - 180x + 940$$

英文读作：The function of x equals nine times the square of x minus one hundred and eighty times x plus nine hundred and forty.

(二) → 趋近于(Approach; tend to)

例 2.51(Example 2.51) $a_n \to \infty$

英文读作：a sub n approaches / tends to infinity.

例 2.52(Example 2.52) $a_n \to a$

英文读作：a sub n approaches / tends to the value of a

(三) lim 极限(Limit)

例 2.53(Example 2.53) $\lim b$

英文读作：the limit of b

(四) ∞ 无穷大(Infinity)

例 2.54(Example 2.54) $\lim\limits_{n\to\infty} S_n = \frac{1}{3}$

英文读作：The limit of S_n as n gets arbitrarily large is one third.

例 2.55(Example 2.55) 上极限 $\lim\limits_{n\to\infty} \sup\ a_n$

英文读作：upper limit of a_n as n approaches infinity

(sup ≡ supra, above)

例 2.56(Example 2.56) 下极限 $\lim\limits_{n\to\infty} \inf\ a_n$

英文读作：lower limit of a_n as n approaches infinity

(iof ≡ infra, below)

例 2.57(Example 2.57)　$\lim\limits_{x \to a} f(x)$

英文读作：limit of $f(x)$ as x approaches a

例 2.58(Example 2.58)　$\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y)$

英文读作：limit of $f(x, y)$（或 the function of x and y）as x approaches a and y approaches b

二、区间(Intervals)

(一) 开区间(Open Interval)

开区间用符号(a, b)表示，设 a，b 为固定数，且 $a < b$. 开区间(a, b)指从 a 到b 端点(不包括 a, b)间的所有实数 x，且 $a < x < b$. 见图 2.1.

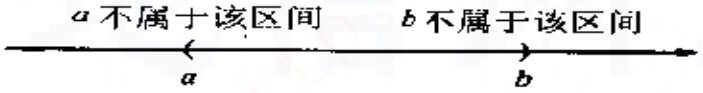


图 2.1

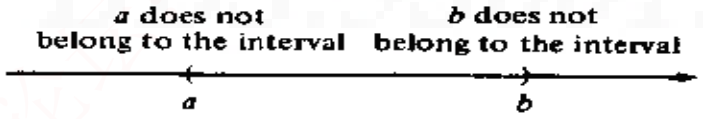


Figure 2.1

The open interval is denoted by (a, b). Let a and b be fixed real numbers with $a < b$. The open interval (a, b) refers to the set of all real numbers x between endpoint a and endpoint b (excluding a and b) with $a < x < b$ (see figure 2.1).

(二) 闭区间(Closed Interval)

闭区间用符号 $[a,b]$ 表示,设 a, b 为固定数,且 $a < b$.闭区间 $[a,b]$ 指从 a 到 b 端点(包括 a,b)间的所有实数 x,且 $a \leqslant x \leqslant b$. 见图 2.2.

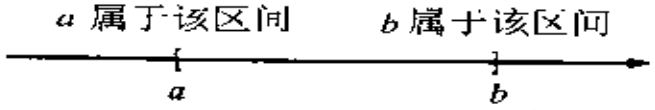


图 2.2

The closed interval is denoted by $[a,b]$. Let a and b be fixed real numbers with $a < b$. The closed interval $[a,b]$ refers to the set of all real numbers x between endpoint a and endpoint b (including a and b) with $a \leqslant x \leqslant b$ (see figure 2.2).

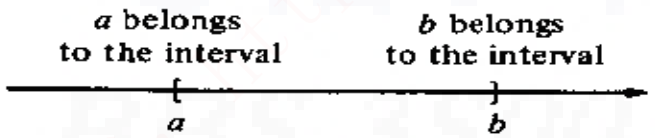


Figure 2.2

(三) 半开区间或半闭区间

(Half-open Interval or Half-closed Interval)

半开区间或半闭区间用符号 $[a,b)$ 或 $(a,b]$ 表示,设 a, b 为固定数,且 $a < b$.半开或半闭区间 $[a,b)$ 指从 a 到 b 端点(包括 a,不包括 b)间的所有实数 x,且 $a \leqslant x < b$;半开或半闭区间 $(a,b]$ 指从 a 到 b 端点(包括 b,不包括 a)间的所有实数 x,且 $a < x \leqslant b$. 见图 2.3 和 2.4.

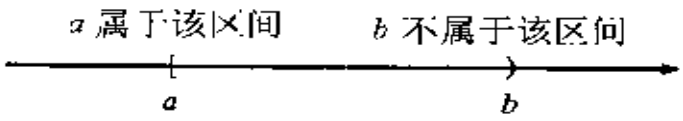


图 2.3

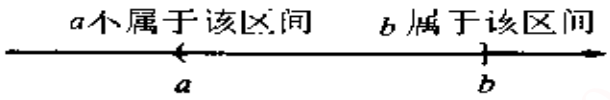


图 2.4

The half-open or half-closed interval is written as $[a, b)$ or $(a, b]$. Let a and b be fixed real numbers with $a < b$. The **half-open or half-closed interval** $[a, b)$ refers to the set of all real numbers x between endpoint a and endpoint b (including a but not b) with $a \leqslant x < b$ (see figure 2.3), while the $(a, b]$ refers to the set of all real numbers between endpoint a and endpoint b (including b but not a) with $a < x \leqslant b$ (see figure 2.4).

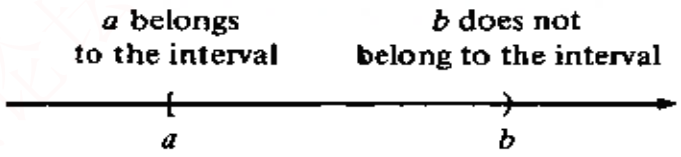


Figure 2.3

例 2.59(Example 2.59) (1) (2,5] (2) $(3, \infty)$

解(Solution):

(1) 区间(2,5]包括 2 和 5 之间的所有实数,包括 5,但不包括 2.用数轴可表示为图 2.5.



Figure 2.4

The interval $(2,5]$ consists of all real numbers between 2 and 5, including 5, but excluding 2 (see figure 2.5).

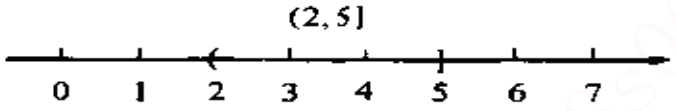


图 2.5(Figure 2.5)

(2) 区间$(3,\infty)$包括大于 3 的所有实数,用数轴可表示为图 2.6.

The interval $(3,\infty)$ consists of all real numbers that are greater than 3 (see figure 2.6).

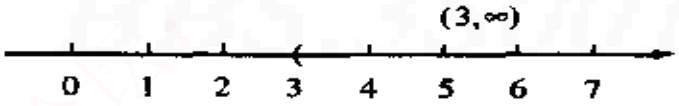


图 2.6(Figure 2.6)

§2.4 微分与积分(Differentials and Integrals)

一、微分(Differentials)

(一) 导数(Derivatives)

$f(x)$　$f(x)$ 的函数(function of x)

$\triangle x$　x 的增量(delta x 或 the increment added to x)

$\triangle f_x$　函数 f_x 的增量(delta f 或 the increment added to function of x)

$\mathrm{d}f(x)$　微分(the differential function f of x)

1. 一阶导数(the First Derivatives)

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

$f'(x)$:　为函数 $f(x)$ 关于 x 的一阶导数

the first derivative of f with respect to x

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

y ':　为 y 的一阶导数

the first derivative of y

$$\dot{\varphi}(t) = \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}$$

$\dot{\varphi}(t)$:　为 φ 的一阶导数

the first derivative of φ

例 2.60　设:　$f(x) = 2x + 1$, 则:　$f'(x) = 2$

Example 2.60　Let $f(x) = 2x + 1$, then $f'(x) = 2$

英文读作：Let function of x equal two times x plus one, then f prime of x *equals two/the first derivative of* f with respect to x equals two.

例 2.61　设：　$y = \frac{1}{x}$，则：　$y' = -\frac{1}{x^2}$

Example 2.61　Let $y = \frac{1}{x}$, then $y' = -\frac{1}{x^2}$

英文读作：Let y equal one over x , then y prime equals minus one over x squared/the first derivative of y with respect to x equals minus one over x squared.

2. 二阶导数(the Second Derivatives)

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$$

$f''(x)$：　为函数 $f(x)$ 关于 x 的二阶导数

the second derivative of f with respect to x .

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

$y''(x)$：　为函数 $y(x)$ 关于 x 的二阶导数

the second derivative of y with respect to x .

$$\ddot{\varphi}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 \varphi(t)}{\mathrm{d}t^2}$$

$\ddot{\varphi}(t)$：　为 $\varphi(t)$ 关于 t 的二阶导数

the second derivative of φ with respect to t .

例 2.62　设 $f(x) = x^3$，则 $f''(x) = (3x^2)' = 6x$.

Example 2.62　Let $f(x) = x^3$, then $f''(x) = (3x^2)' = 6x$.

英文读作：Let function of x equal x to the third power, then the second derivative of f with respect to x equals the first derivative of three times x squared equals six times x .

例 2.63　设 $y = e^{-\alpha x}$，则 $\frac{d^2y}{dx^2} = \alpha^2 e^{-\alpha x}$.

Example 2.63　Let $y = e^{-\alpha x}$, then $\frac{d^2y}{dx^2} = \alpha^2 e^{-\alpha x}$.

英文读作：Let y equal e to the power of minus alpha times x , then the second derivative of y with respect to x equals alpha squared times e to the power of minus alpha times x .

3．n 阶导数(the nth Derivatives)

$$f^{n}(x) = \frac{d^n f(x)}{dx^n}$$

$f^{(n)}(x)$：　为 $f(x)$ 关于 x 的 n 阶导数

the nth derivative of f with respect to x .

$$y^{(n)} = \frac{d^n y}{dx^n}$$

$y^{(n)}$：　为 y 关于 x 的 n 阶导数

the nth derivative of y with respect to x .

例 2.64(Example 2.64)　$a_3y''' + a_2y'' + a_1y' + a_0y = 0$

英文读作：a sub three times the third derivative of the function y with respect to x plus a sub two times the second derivative of the function y with respect to x plus a sub one times derivative of y with respect to x plus a sub zero times y equals zero.

或：a sub three times y triple prime plus a sub two times y double

prime plus a sub one times y prime plus a sub zero times y equals zero.

(二) 偏导数(Partial Derivatives)

有两个或两个以上自变量的函数称为多元函数,例 $f(x,y)$ 称为二元函数.多元函数关于自变量的导数称为**偏导数**.

The functions of two or more independent variables are called multivariate functions, such as $f(x,y)$, which is a function of two variables. The derivatives of functions of several variables are known as **partial derivatives**.

f_x 或 $\frac{\partial f}{\partial x}$: 为 $f(x,y)$ 关于 x 的一阶偏导

the first partial derivative of f with respect to x

f_y 或 $\frac{\partial f}{\partial y}$: 为 $f(x,y)$ 关于 y 的一阶偏导

the first partial derivative of f with respect to y

f_{xx} 或 $\frac{\partial^2 f}{\partial x^2}$: 为 $f(x,y)$ 关于 x 的二阶偏导

the second partial derivative of f with respect to x

f_{xy} 或 $\frac{\partial^2 f}{\partial x^2 y}$: 为 $f(x,y)$ 关于 x、y 的二阶混合偏导

the second mixed partial derivative of f with respect to x and y

f_{yy} 或 $\frac{\partial^2 f}{\partial y^2}$: 为 $f(y,y)$ 关于 y 的二阶偏导

the second partial derivative of f with respect to y

例 2.65 设: $z = x^3y^3 - 3xy^3 + xy + 1$

求 (1) $\frac{\partial z}{\partial x}$ (2) $\frac{\partial z}{\partial y}$

(3) $\frac{\partial^2 z}{\partial x^2}$ (4) $\frac{\partial^2 z}{\partial y \partial x}$

(5) $\frac{\partial^2 z}{\partial y^2}$ (6) $\frac{\partial^3 z}{\partial x^3}$

解：(1) $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 + y$

(2) $\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 + x$

(3) $\frac{\partial^2 z}{\partial x^2} = 6xy^2$

(4) $\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 + 1$

(5) $\frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy$

(6) $\frac{\partial^2 z}{\partial x^3} = 6y^2$

Example 2.65 Let $z = x^3y^3 - 3xy^3 + xy + 1$

Find

(1) $\frac{\partial z}{\partial x}$ (the partial derivative of z with respect to x)

(2) $\frac{\partial z}{\partial y}$ (the partial derivative of z with respect to y)

(3) $\frac{\partial^2 z}{\partial x^2}$ (the second partial derivative of z with respect to x)

(4) $\frac{\partial^2 z}{\partial y \partial x}$ (the partial derivative of z with respect to x of the partial derivative of z with respect to y)

(5) $\frac{\partial^2 z}{\partial y^2}$ (the second partial derivative of z with respect to y)

(6) $\frac{\partial^2 z}{\partial x^3}$ (the third partial derivative of z with respect to x)

Solution: (1) $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 + y$

(2) $\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 + x$

(3) $\frac{\partial^2 z}{\partial x^2} = 6xy^2$

(4) $\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 + 1$

(5) $\frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy$

(6) $\frac{\partial^2 z}{\partial x^3} = 6y^2$

(三) 全微分(Complete Differentials)

对于一个二元函数 $z(x,y)$,全微分记为

$$dz(x,y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For a function of two variables $z(x,y)$, the complete differential is denoted by

$$dz(x,y) = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

英文读作: The differential of z with respect to x and y equals the partial derivative of z with respect to x times the differential of x plus the partial derivative of z with respect to y times the differential of y.

对于一个三元函数 $u(x,y,z)$,全微分记为

$$du(x,y,z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

For a function of three variables $u(x,y,z)$, the complete differential is denoted by

$$du(x,y,z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

英文读作：The differential of u with respect to x, y and z equals the partial derivative of u with respect to x times the differential of x plus the partial derivative of u with respect to y times the differential of y plus the partial derivative of u with respect to z times the differential of z .

对于 n 元函数 $f(x_1, x_2, \cdots x_n)$，则全微分记为

$$\mathrm{d}f(x_1, x_2, \cdots x_n) = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n$$

For a function of n variables $f(x_1, x_2, \cdots x_n)$, the complete differential is denoted by

$$\mathrm{d}f(x_1, x_2, \cdots x_n) = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n$$

英文读作：The differential of f with respect to x_1, x_2 up to x_n equals the partial derivative of f with respect to x_1 times the differential of x_1 plus the partial derivative of f with respect to x_2 times the differential of x_2 plus up to plus the partial derivative of f with respect to x_n times the differential of x_n .

例 2.66　计算　$z = x^2y + y^2$ 的全微分 .

解：　$\because \quad \dfrac{\partial z}{\partial x} = 2xy, \quad \dfrac{\partial z}{\partial y} = x^2 + 2y$

$\therefore \quad \mathrm{d}z = 2xy\mathrm{d}x + (x^2 + 2y)\mathrm{d}y$

Example 2.66　Find the complete differential for $z = x^2y + y^2$.

Solution　Since　$\dfrac{\partial z}{\partial x} = 2xy, \quad \dfrac{\partial z}{\partial y} = x^2 + 2y$

thus　$\mathrm{d}z = 2xy\mathrm{d}x + (x^2 + 2y)\mathrm{d}y$

英文读作：The differential of z equals two times x times y times the differential of x plus the quantity x squared plus two times y times the differential of y .

二、积分(Integrals)

(一) 不定积分(Indefinite Integrals)

$$\int f(x)\mathrm{d}x$$

$f(x)$ 关于 x 的不定积分,其中$\int$ 叫做积分号,函数 $f(x)$ 叫做被积函数,$\mathrm{d}x$ 是关于 x 的微分,而 x 叫做积分变量.

The indefinite integral of a function f with respect to x, where the symbol $\int$ is called an integral sign, function $f(x)$ the integrand, $\mathrm{d}x$ the differential of x and x is the variable of the integral.

例 2.67(Example 2.67) $\int ax\ \mathrm{d}x = a\int x\ \mathrm{d}x = \frac{ax^2}{2} + C$,
其中 C 称为积分常数.

英文读作: The indefinite integral of a times x with respect to x equals a times the indefinite integral of x with respect to x equals the quantity a times x squared over two plus C, which is the constant of the integral.

或: The indefinite integral of $ax\mathrm{d}x$ equals a times the indefinite integral of $x\mathrm{d}x$ equals ax^2 over 2 plus C, which is the constant of the integral.

例 2.68(Example 2.68) $\int (2x + 3)\mathrm{d}x = x^2 + 3x + C$

英文读作: The indefinite integral of the quantity two times x plus three with respect to x equals x squared plus 3 times x plus C.

例 2.69(Example 2.69) $\int \frac{a}{x - a}\ \mathrm{d}x = a\log|x - a| + C$

英文读作：The indefinite integral of the quantity a over x minus α with respect to x equals a times the quantity log of the absolute value of x minus α plus C .

(二) 定积分(Definite Integrals)

$$\int_a^b f(x)\mathrm{d}x = [g(x)]_a^b$$

函数$f(x)$从a到b的定积分. $g(x)$称为积分函数,(a,b)称为积分区间.

英文读作：The definite integral of (the function) f of x with limits a and b (from a to b or from $x = a$ to $x = b$) with respect to x equals the difference of g of x evaluated at b and a (the definite integral of f, a to b , equals g, a to b). $g(x)$ is the function of the integral and (a,b) is called an integral interval.

例 2.70(Example 2.70) $\int_0^4 12{,}000\sqrt{t}\ \mathrm{d}t$

英文读作：the definite integral of twelve thousand times the square root of t from zero to four (with respect to t)

或：the definite integral of twelve thousand times the square root over the interval $[0,4]$ with respect to t

例 2.71(Example 2.71) $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\alpha\cos x}$

英文读作：the integral from zero to pi over two of the quantity one over one plus α times cosine of x with respect to x

或：the integral from zero to pi over two of dx over one plus α times cosine of x (with respect to x)

(三) 重积分(Double Integrals)

$\iint f(x,y)\,\mathrm{d}x\mathrm{d}y$　$f(x,y)$ 关于 x,y 的重积分.

英文读作：The double integral of f of x,y (with respect to x and y).

重积分还可记为

$\iint_R f(x,y)\,\mathrm{d}A$　$f(x,y)$ 在 R 区域的关于 x,y 的重积分.

或　$\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

英文读作：The double integral of f over (region) R with respect to x and y.

或：integrating from a to b of f of x,y with respect to x and from c to d with respect to y.

例 2.72(Example 2.72)　$\int_{-2}^{3}\int_{-2}^{1} xy\,\mathrm{d}y\mathrm{d}x$

英文读作：integrating from -2 to 1 of x,y with respect to y and from -2 to 3 with respect to x

例 2.73(Example 2.73)　$\int_{1}^{2}\int_{0}^{1}(2x+y)\,\mathrm{d}x\mathrm{d}y$

英文读作：integrating from zero to one with respect to x and from one to two of the quantity two times x plus y with respect to y

(四) 三重积分(Triple Integrals)

$\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

表示函数 $f(x,y,z)$ 在区域 Ω 的三重积分. 三重积分还可记为

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}v$$

英文读作：the triple integral of f of x,y,z with respect to x ,y and $z$over the region Ω /the triple integral of f over (region) Ω

例 2.74(Example 2.74)

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}v = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

英文读作：The triple integral of f over region Ω equals integrating from $z_1(x,y)$ to $z_2(x,y)$ with respect to z , from $y_1(x)$ to $y_2(x)$ with respect to y and from a to b with respect to x .

(五) 闭路积分(Closed Linear Integrals)

$\oint_c (x\mathrm{d}y - y\mathrm{d}x)$ 表示闭路积分，$\oint$ 表示闭路积分符号，c 为积分区域的边界曲线.

$\oint (x\mathrm{d}y - y\mathrm{d}x)$ represents a closed linear integral in which $\oint$ is the sign of the closed linear integral and c the boundary curve in the integral region.

§2.5 指数和对数(Exponents and Logarithms)

一、指数函数(Exponential Functions)

$$f(x) = a^x$$

表示以 a 为底的指数函数，a 为大于 1 的常数，x 为指数 .

The function f of x is an exponential function with base a , where a

is a positive constant greater than 1 and x is the exponent.

$\exp(x) = e^x$　表示以 e 为底的 x 的指数函数 .

Exponential functions are also denoted by EXP or exp; for instance, $\exp(x) = e^x$ means the exponential function of x with base e.

例 2.75(Example 2.75)　$a^{-n} = \frac{1}{a^n}$

英文读作：a to the minus nth power equals one over a to the nth power.

例 2.76(Example 2.76)　$a^0 = 1$　　$(a \neq 0)$

英文读作：a to the zero power equals one, where a is not equal to zero.

例 2.77(Example 2.77)

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad (a \geqslant 0,\ m, n \text{ 均为正整数})$$

英文读作：a to the m over n power equals the nth root of a to the mth power, where a equals or is greater than zero, and m and n are both positive integers.

例 2.78(Example 2.78)

$$a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}} \qquad (a > 0,\ m, n \text{ 均为正数})$$

英文读作：a to the minus m over n power equals one over the nth root of a to the mth power, where a is greater than zero, and m and n are both positive numbers.

例 2.79(Example 2.79)　$f(x) = \frac{e^x + e^{-x}}{2}$

英文读作：The function f of x equals e to the xth power plus e to the minus xth power over two.

例 2.80(Example 2.80)　$S = P\left(1 + \frac{r}{n}\right)^{nt}$

英文读作：S equals P times the quantity one plus r over n to the power of n times t .

二、对数函数(Logarithmatic Functions)

(一) 常用对数(Common Logarithms)

以 10 为底的对数称为常用对数，可标记为"$\log_{10}$"，常缩写为"$\log x$"或"$\lg x$"，其函数形式为

$$\log x = \log_{10} x \qquad \text{其中 } x > 0.$$

The logarithm with base 10 is called a common logarithm, and the symbol "$\log x$"or "$\lg x$" is often used as an abbreviation for $\log_{10}$. Thus, the common logarithmstic function is defined by

$$\log x = \log_{10} x \qquad \text{for } x > 0.$$

英文读作：Log of x equals the logarithm of x with base 10, for x is greater than zero.

例 2.81(Example 2.81)　$\log R = A + \frac{B}{T} + \frac{K}{\log R}$

英文读作：Log R equals A plus B over T plus K over log$R$.

例 2.82(Example 2.82)　$\log N^k = K \log N$

英文读作：The logarithm (log) of the kth power of N equals K times the logarithm of N .

例 2.83(Example 2.83)　$\log \sqrt[q]{N} = \frac{1}{q}\log N$

英文读作：The logarithm of the qth root of N equals the logarithm of N divided by q (equals one over q times $\log N$).

例 2.84(Example 2.84)　$\log A^{\frac{1}{n}} = \frac{1}{n}\log A$

英文读作：The log of A to the one over n power equals one over n times the log of A

(二) 自然对数(Natural Logarithms)

以 e(e ≈ 2.718)为底的对数称为自然对数，可标记为“$\log_e x$”，常缩写为“ln x”，其函数形式为

$$\ln x = \log_e x \qquad 当\ x > 0.$$

Logarithms with base e are called natural logarithms and the symbol "ln x" is often used as an abbreviation for $\log_e x$. Thus, the natural logarithmatic function is defined by

$$\ln x = \log_e x \qquad \text{for } x > 0.$$

英文读作：Ell en x equals the logarithm to the base e of x, for x is greater than zero.

例 2.85　用有 ln 键的计算器计算：

(1) ln 7,124　　(2) ln 0.05319

解：　在十位数字计算器上得到

(1) ln 7,124 = 8.871224644

(2) ln 0.05319 = −2.933884870

Example 2.85　Use a calculator with an ln key to evaluate

(1) ln 7,124　　(2) ln 0.05319

Solution　On a 10-digit calculator, we obtain

(1) ln 7,124 = 8.871224644

英文读作：Ell en seven thousand one hundred and twenty-four equals eight point eight seven one double two four six double four.

(2) $\ln 0.05319 = -2.933884870$

英文读作：Ell en zero point zero five three one nine equals minus two point nine double three double eight four eight seven zero.

例 2.86(Example 2.86)　$\ln e^{\sqrt{5}} = \sqrt{5}$

英文读作：Ell en e to the square root of five equals the square root of five.

例 2.87(Example 2.87)　$f(x) = 1 + \ln(x - 2)$

英文读作：The function f of x equals one plus ell en the quantity x minus two.

例 2.88(Example 2.88)　$h(x) = \ln \dfrac{x}{x + 1}$

英文读作：The function h of x equals ell en x over x plus one.

例 2.89(Example 2.89)　(1) $\log_b x = \dfrac{\log x}{\log b}$　　(2) $\log_b a = \dfrac{1}{\log_a b}$

英文读作：(1) The logarithm to the base b of x equals the logarithm of x over the logarithm of b ;

(2) The logarithm to the base b of a equals one over the logarithm to the base a of b .

§2.6 三角函数、反三角函数、双曲函数和反双曲函数 (Trigonometric, Inverse Trigonometric, Hyperbolic and Inverse Hyperbolic Functions)

一、三角函数(Trigonometric Functions)

(一) sin 正弦(sine)

例 2.90(Example 2.90) $\sin\frac{\pi}{2} = 1$

英文读作: The sine of pi over two equals one.

例 2.91(Example 2.91) $\sin(-t) = -\sin t$

英文读作: The sine of minus t equals the minus sine of t .

例 2.92(Example 2.92) $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$

英文读作: The sine of three theta is equivalent to three times the sine of theta minus four times the quantity sine theta cubed.

例 2.93(Example 2.93) $\sin(t + 2\pi n) = \sin t$

英文读作: The sine of the quantity t plus two times pi times n equals the sine of t .

(二) cos 余弦(cosine)

例 2.94(Example 2.94) $\cos\frac{\pi}{2} = 0$

英文读作: The cosine of pi over two equals zero.

例 2.95(Example 2.95)　$\cos t = -\sqrt{1-\sin^2 t}$

英文读作：The cosine of t equals minus the square root of one minus the quantity sine t squared.

例 2.96(Example 2.96)

$$\sin A + \sin B \equiv 2\sin\frac{1}{2}(A+B)\cdot\cos\frac{1}{2}(A-B)$$

英文读作：The sine of A plus the sine of B is equivalent to two times the sine of one-half the sum of A and B times the cosine of one-half of the difference of A and B .

例 2.97(Example 2.97)　$\sin\frac{1}{2}A \equiv \pm\sqrt{\frac{1-\cos A}{2}}$

英文读作：The sine of one-half of A is equivalent to plus or minus the square root of the quantity one minus the cosine of A over two.

(三) tan　正切(tangent)

例 2.98(Example 2.98)　$\tan\frac{\pi}{4} = 1$

英文读作：The tangent of pi over four equals one.

例 2.99(Example 2.99)　$\tan t = \frac{y}{x}$

英文读作：The tangent of t equals y over x .

例 2.100(Example 2.100)　$\tan t = \frac{\sin t}{\cos t}$

英文读作：The tangent of t equals the sine of t over the cosine of t .

例 2.101(Example 2.101)　$\tan\theta = \frac{-3}{-1} = 3$

英文读作：The tangent of theta equals minus three over minus one equals three.

(四) cot　余切(cotangent)

例 2.102(Example 2.102)　$\cot\frac{\pi}{6} = \sqrt{3}$

英文读作：The cotangent of pi over six equals the square root of three.

例 2.103(Example 2.103)　$\cot t = \frac{x}{y} = \frac{\cos t}{\sin t}$

英文读作：The cotangent of t equals x over y equals the cosine of t over the sine of t .

例 2.104(Example 2.104)　$\cot\theta = \frac{\text{adj}}{\text{opp}}$

英文读作：The cotangent of θ equals the adjacent side over the opposite side.

例 2.105(Example 2.105)　$\cot\theta = -\frac{17}{15}$

英文读作：The cotangent of theta equals minus seventeen over fifteen.

(五) sec　正割(secant)

例 2.106(Example 2.106)　$\sec 0 = 1$

英文读作：The secant of zero equals one.

例 2.107(Example 2.107)　$\sec\alpha = \frac{1}{\cos\alpha}$

英文读作：The secant of alpha equals one over the cosine of alpha.

例 2.108(Example 2.108)　$\sec \frac{\pi}{4} = \sqrt{2}$

英文读作：The secant of pi over four equals the square root of two.

(六) csc　余割(cosecant)

例 2.109(Example 2.109)　$\csc\pi = \infty$

英文读作：The cosecant of pi equals infinity.

例 2.110(Example 2.110)　$\csc^2\alpha - \operatorname{ctg}^2\alpha = 1$

英文读作：The cosecant alpha squared minus the cotangent alpha squared equals one.

例 2.111(Example 2.111)　$\csc t = \frac{1}{\sin t}$

英文读作：The cosecant of t equals one over the sine of t.

二、反三角函数(Inverse Trigonometric Functions)

(一) arcsin 或 $\sin^{-1}$　反正弦
(the inverse sine function 或 arcsine function)

例 2.112(Example 2.112)　$-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

或　$-\frac{\pi}{2} \leq \arcsin^{-1}x \leq \frac{\pi}{2}$

英文读作：The arcsine function of x is equal to or greater than minus pi over two, and is equal to or less than pi over two.

例 2.113(Example 2.113)　$\arcsin(-\frac{1}{2}) = -\pi/6$

英文读作：The arcsine / the inverse sine of minus one-half equals minus the quantity pi over six.

（二）arccos 或 $\cos^{-1}$　反余弦

（the inverse cosine function 或 the arccosine function）

例 2.114（Example 2.114）　$\arccos x = \sqrt{1 - x^2}$

英文读作：The arccosine of x equals the square root of the quantity one minus x squared.

例 2.115（Example 2.115）　$\sin^{-1}(-1) - \cos^{-1}(-1)$

英文读作：the inverse sine of minus one minus inverse cosine of minus one

或：the arcsine of minus one minus the arccosine of minus one

（三）arctan 或 $\tan^{-1}$　反正切

（the inverse tangent function 或 the arctangent function）

例 2.116（Example 2.116）　$\arctan x = \dfrac{x}{\sqrt{1 + x^2}}$

英文读作：The arctangent of x equals x over the square root of the sum of one plus x squared.

例 2.117（Example 2.117）　$\dfrac{1}{2}y = \tan^{-1}\sqrt{\dfrac{1 - x}{1 + x}}$

英文读作：One-half y equals the arctangent of the square root of the quantity one minus x over one plus x .

（四）arccot 或 $\cot^{-1}$　反余切

（the inverse cotangent 或 the arccotangent）

例 2.118（Example 2.118）　$\operatorname{arccot} x = \dfrac{x}{\sqrt{1 + x^2}}$

英文读作：The cosine of the arccotangent of x equals x over the square

root of the quantity of one plus x squared.

例 2.119(Example 2.119)　$\arctan\alpha + \operatorname{arccot}x = \frac{\pi}{2}$

英文读作：The arctangent of α plus the arccotangent of x equals pi over two.

(五) arcsec 或 $\sec^{-1}$　反正割
(the inverse secant 或 the arcsecant)

例 2.120(Example 2.120)　$\operatorname{arcsec}x$

英文读作：the inverse secant of x 或 the arcsecant of x

(六) arccsc　或 $\csc^{-1}$　反余割
(the inverse cosecant 或 the arccosecant)

例 2.121(Example 2.121)　$\operatorname{arcsec}x + \operatorname{arccsc}x = \frac{\pi}{2}$

英文读作：The arcsecant of x plus the arccocecant of x equals pi over two.

三、双曲函数(Hyperbolic Functions)

(一) sinh 或 sh　双曲正弦(the hyperbolic sine)

例 2.122(Example 2.122)　$\operatorname{sh}x = \frac{e^x - e^{-x}}{2}$

英文读作：The hyperbolic sine of x equals the quantity e to the xth power minus e to the minus xth power over two.

(二) cosh 或 ch　双曲余弦(the hyperbolic cosine)

$$\operatorname{ch}x = \frac{e^x + e^{-x}}{2}$$

英文读作：The hyperbolic cosine of x equals the quantity e to the xth power plus e to the minus xth power over two.

例 2.123(Example 2.123)　$\mathrm{ch}^2x - \mathrm{sh}^2x = 1$

英文读作：The hyperbolic cosine of x squared minus the hyperbolic sine of x squared equals one.

(三) tanh 或 th　双曲正切(the hyperbolic tangent)

$$\mathrm{th}x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

英文读作：The hyperbolic tangent of x equals the quantity e to the xth power minus e to the minus xth power over e to the xth power plus e to the minus xth power.

(四) coth 或 cth　双曲余切(the hyperbolic cotangent)

例 2.124(Example 2.124)　$\mathrm{cth}x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

英文读作：The hyperbolic cotangent of x equals the quantity e to the x th power plus e to the minus xth power over e to the xth power minus e to the minus xth power.

例 2.125(Example 2.125)　$1 - \mathrm{th}^2x = \frac{1}{\mathrm{ch}^2x}$

英文读作：One minus the hyperbolic tangent x squared equals one over the hyperbolic cosine x squared.

(五) sech　双曲正割(the hyperbolic secant)

$$sechx = \frac{1}{\mathrm{ch}x}$$

英文读作：The hyperbolic secant of x equals one over the hyperbolic

cosine of x .

(六) csch　双曲余割(the hyperbolic cocecant)

$$\operatorname{csch} x = \frac{1}{\operatorname{sh} x}$$

英文读作：The hyperbolic cocecant of x equals one over the hyperbolic sine of x .

四、反双曲函数(Inverse Hyperbolic Functions)

(一) arcsinh 或 $\sinh^{-1}$, sh^{-1}　反双曲正弦
(the inverse hyperbolic sine 或 arc-hyperbolic sine)

例 2.126(Example 2.126)　$\mathrm{sh}^{-1} x = \ln(x \pm \sqrt{x^2 + 1})$

英文读作：The arc-hyperbolic sine of x equals ell en the quantity x plus or minus the square root of x squared plus one.

(二) arccosh 或 $\cosh^{-1}$, ch^{-1}　反双曲余弦
(the inverse hyperbolic cosine 或 arc-hyperbolic cosine)

例 2.127(Example 2.127)　$\mathrm{ch}^{-1} x = \ln(x \pm \sqrt{x^2 - 1})$

英文读作：The arc-hyperbolic cosine of x equals ell en the quantity x plus or minus the square root of x squared minus one.

(三) arctanh 或 $\tanh^{-1}$, th^{-1}　反双曲正切
(the inverse hyperbolic tangent / arc-hyperbolic tangent)

例 2.128(Example 2.128)　$\mathrm{th}^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}$

英文读作：The arc-hyperbolic tangent of x equals one-half times ell en the quantity one plus x over one minus x .

(四) arccoth 或 $\coth^{-1}$, cth^{-1}　反双曲余切

(the inverse hyperbolic cotangent / arc-hyperbolic cotangent)

例 2.129(Example 2.129)　$\mathrm{cth}^{-1}x = \frac{1}{2}\ln\frac{x+1}{x-1}$

英文读作：The arc-hyperbolic cotangent of x equals one-half times ell en the quantity x plus one over x minus one.

§2.7　集合(Sets)

一、集合(Sets)

(一) $A = \{a, b, c, d, e\}$

A 是由元素 a, b, c, d, e 组成的集合.

A is a set consisting of a, b, c, d and e.

例 2.130(Example 2.130)　$K = \{1, 2, 3, 4, 5\}$

英文读作：K equals the set whose members(elements) are one, two, three, four, five.

例 2.131(Example 2.131)　$\{0, 1, 2, \cdots, 100\}$

英文读作：the set whose members are the natural numbers from zero, one, two through one hundred

例 2.132(Example 2.132)

$$I_1 = \{0, 1, -1, 2, -2, \cdots, n, -n\}$$

英文读作：I sub one equals the set whose elements are zero, one, minus one, two, minus two, etc., till n, minus n.

例 2.133(Example 2.133)　$S = \{x \mid 5 < x < 7\}$

英文读作：The set S is the set of numbers x such that x is between five and seven.

例 2.134(Example 2.134)　$\{x \mid x < 5\}$

英文读作：the set whose members are all x that satisfy the condition that x is less than five

或：the set of all x such that x is less than five

例 2.135(Example 2.135)　$\{(x, y) \mid x > y\}$

英文读作：the set whose members are all ordered pairs x, y that satisfy the condition that x is greater than y

例 2.136　全体自然数的集合通常简称为自然数集，记作 **N**.

Example 2.136　The set of all the natural numbers is usually called a natural numerical set, denoted by **N**.

例 2.137　全体整数的集合通常简称为整数集，记作 **Z**.

Example 2.137　The set of all integers is usually called an integer set, denoted by **Z**.

例 2.138　全体有理数的集合通常简称为有理数集，记作 **Q**.

Example 2.138　The set of all rational numbers is usually called a rational number set, denoted by **Q**.

例 2.139　全体实数的集合通常简称为实数集，记作 **R**.

Example 2.139　The set of all real numbers is usually called a real number set, denoted by **R**.

(二) $\in$　属于(a member of set 或 an element of set)

$\notin$ / $\ni$　不属于(not a member of set 或 not an element of set)

例 2.140(Example 2.140)　$N = \{1,2,3,\cdots\}$.

(1) $4 \in N$　(2) $6 \in N$　(3) $4.5 \notin N$

(4) $0 \notin N$　(5) $-1 \notin N$

英文读作: For the set N whose members are one, two, three, etc.

(1) Four is a member of N;

(2) six is a member of N;

(3) four point five is not a member of N;

(4) zero is not a member of N;

(5) minus one is not a member of N.

二、空集、子集、交集、并集、补集

(Empty Sets, Subsets, Intersections, Unions and Complements)

(一) { }; ϕ　空集(Empty Sets)

例如 $L = \phi$　L 是一空集

For example, $L = \phi$.　L is an empty set.

(二) $\subseteq$　是……的子集;包含于(is a subset of)

$A \subseteq B$　A 是 B 的子集,读作"A 包含于 B".

英文读作: Set A is a subset of set B, read "A is included in B".

例 2.141　看下列集合:

$U = \{1,2,3,4,5,6,7,8,9\}$

$A = \{2,4,6,8\}$　$B = \{1.3,5,7,9\}$　$C = \{5,7\}$

则: $A \subseteq U$, $B \subseteq U$, $C \subseteq U$, $C \subseteq B$, 但 C 不是 A 的子集,因为 $5 \in C, 5 \notin A$.

Example 2.141 Consider the following sets:

$U = \{1,2,3,4,5,6,7,8,9\}$

$A = \{2, 4, 6, 8\}$ $B = \{1,3,5,7.9\}$ $C = \{5,7\}$

Here, $A \subseteq U$, $B \subseteq U$, $C \subseteq U$ and $C \subseteq B$, but C is not a subset of A since $5 \in C$ and $5 \notin A$.

英文读作: U equals the set whose members are one, two, three, four, five, six, seven, eight and nine;

A equals the set whose members are two, four, six and eight;

B equals the set whose members are one, three, five, seven and nine;

C equals the set whose members are five and seven.

Here, A is a subset of U, B is a subset of U, C is a subset of U and a subset of B, but not a subset of A since 5 is a member of C and not a member of A.

描述集合之间关系常用的方法是用矩形图代表通集,用圆形区域代表各子集.

A useful way to depict relationships among sets is to let the universal set be represented by a rectangle, with each subset in the universe represented by a circular region.

例 2.142 $A \subseteq B$, A 是 B 的子集,或 A 包含于 B. 见图 2.7.

Example 2.142 $A \subseteq B$, A is a subset of B or A is part of B (see figure 2.7).

例 2.143 $B \subseteq A$, B 是 A 的子集,或 B 包含子 A. 见图 2.8.

Example 2.143 $B \subseteq A$, B is a subset of A or B is part of A (see figure 2.8).

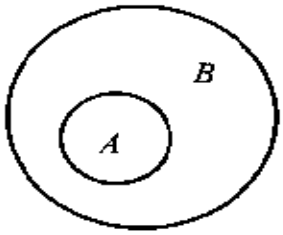


图 2.7(Figure 2.7)

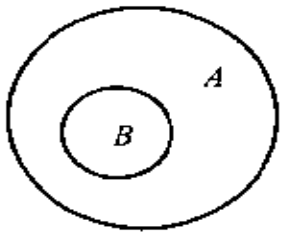


图 2.8(Figure 2.8)

例 2.144　若 $A \subseteq B$，$B \subseteq A$，则 $A = B$（图 2.9）.

Example 2.144　If $A \subseteq B$, $B \subseteq A$, then $A = B$ (see figure 2.9).

例 2.145　若 $A \subseteq B, B \subseteq C$，则 $A \subseteq C$，见图 2.10.

Example 2.145　If $A \subseteq B, B \subseteq C$, then $A \subseteq C$ (see figure 2.10).

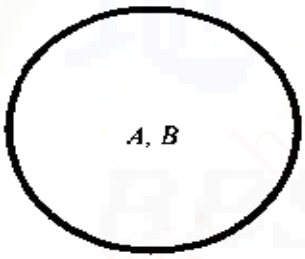


图 2.9(Figure 2.9)

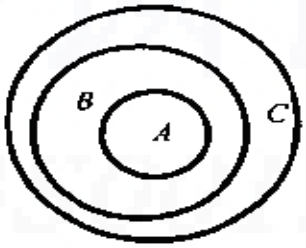


图 2.10(Figure 2.10)

例 2.146　$\phi \subseteq A$，空集是 A 的子集，空集包含于 A.

Example 2.146　An empty set is a subset of A or an empty set is part of A .

例 2.147　$A \subseteq A$，A 是自身的子集.

Example 2.147　A is a subset of set A itself.

例 2.148　若 $A \subseteq B$，但 $B \subseteq A$ 不成立，即 $A \neq B$，则 A 为 B 的一个真子集.

Example 2.148　If A is a subset of B , but B is not a subet of A , that is, A is not equal to B, then A is a real subset of B.

(三) 交集(Intersections)

$A \cap B$　由属于 A 且属于 B 的元素组成的集合，读作 A 交 B，见图 2.11.

The intersection, denoted by $A \cap B$, is the set consisting of all elements common to A and B (see figure 2.11).

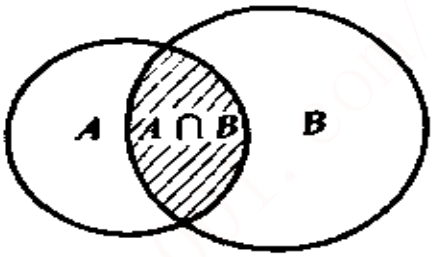


图 2.11(Figure 2.11)

例 2.149(Example 2.149)　$A \cap A = A$

英文读作：The set consisting of all elements common to A equals A itself.

例 2.150(Example 2.150)　$A \cap \phi = \phi$

英文读作：The set consisting of all elements common to A and empty set ϕ equals ϕ .

例 2.151(Example 2.151)　$A \cap B = B \cap A$

英文读作：The set consisting of all elements common to A and B equals the set consisting of all elements common to B and A .

(四) ∪　并集；合集(Unions)

$A \cup B$　由属于 A 或属于 B 的元素组成的集合，叫做 A、B 的并集，读作“A 并 B”，见图 2.12.

The union, denoted by $A \cup B$, is the set consisting of all elements common to A and B (see figure 2.12).

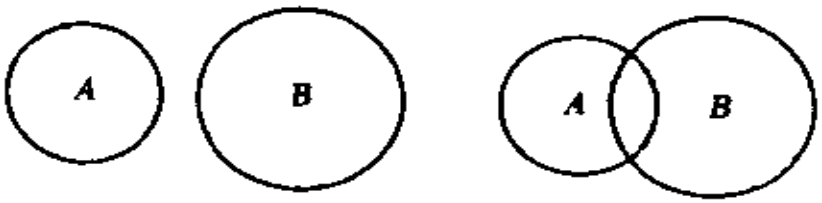


图 2.12(Figure 2.12)

例 2.152　若　$A = \{x,2,3\}, B = \{y,4,3,6\}$,
　　则　$A \cup B = \{3\}$. 如图 2.13.

Example 2.152　If $A = \{x,2,3\}, B = \{y,4,3,6\}$,
then $A \cup B = \{3\}$ (see figure 2.13).

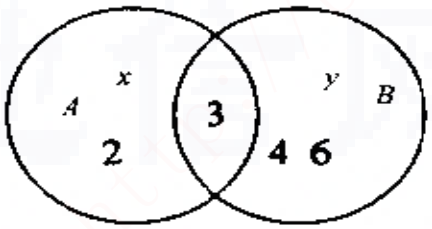


图 2.13(Figure 2.13)

例 2.153　若　$A = \{1,2,4\}, B = \{5,7,8\}$,
　　则　$A \cup B = \phi$. 如图 2.14.

Example 2.153　If $A = \{1,2,4\}, B = \{5,7,8\}$,
then $A \cup B = \phi$ (see figure 2.14).

例 2.154(Example 2.154)

若(If) $A \cup B = B \cup A$,则(then) $A = B$.

英文读作: If the set consisting of all elements of A or B or both equals the set consisting of all elements of B or A or both, then set

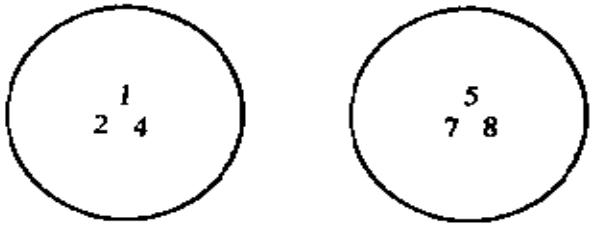


图 2.14(Figure 2.14)

A equals set B .

例 2.155(Example 2.155)

由图 2.15(From figure 2.15),

若(If) $A = \{2,3,5,6\}$

$B = \{3,4,6,7\}$

$C = \{5,6,7,8\}$

则(then): $A \cup C = \{5,6\}$

$B \cup C = \{6,7\}$

$A \cup B = \{3,6\}$

$(A \cup B) \cup C = \{6\}$

$A \cup (B \cup C) = \{6\}$

$\therefore \quad (A \cup B) \cup C = A \cup (B \cup C)$

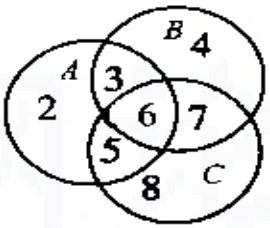


图 2.15(Figure 2.15)

(五) 补集(Complements)

$\bar{A} = \{x \mid x \in I$,且 $x \notin A\}$.

由补集定义容易知道,对于任何集合 A,有

$A \cup \bar{A} = I$, $A \cap \bar{A} = \phi$, $\bar{\bar{A}} = A$,

其中 $\bar{\bar{A}}$ 表示 $\bar{A}$ 在 I 中的补集(见图2.16).

From the definition above, we may know that for any set A , there is

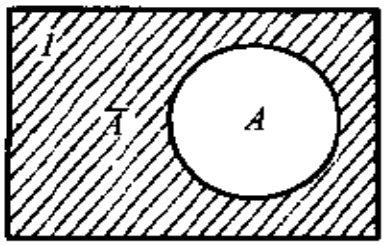


图 2.16(Figure 2.16)

$$A \cup \bar{A} = I,\ A \cap \bar{A} = \phi,\ \bar{\bar{A}} = A,$$

where $\bar{A}$ indicates the complement of I, denoted by $\bar{A}$ (figure 2.16).

例 2.156　如果 $I = \{1,2,3,4,5,6\}$, $A = \{1,3,5\}$, 那么

$$\bar{A} = \{2,4,6\}$$

容易看出

$$\{1,3,5\} \cup \{1,2,3,4,5,6\} = I$$
$$\{1,3,5\} \cap \{2,4,6\} = \phi$$
$$\bar{\bar{A}} = \{1,3,5\} = A$$

Example 2.156　If $I\{1,2,3,4,5,6\}$, $A = \{1,3,5\}$, then

$$\bar{A} = \{2,4,6\}.$$

It is easy to see

$$\{1,3,5\} \cup \{1,2,3,4.5,6\} = I,$$
$$\{1,3,5\} \cap \{2,4,6\} = \phi,$$
$$\bar{\bar{A}} = \{1,3,5\} = A.$$

例 2.157　设 $I = \{1,2,3,4,5,6,7,8\}$, $A = \{3,4,5\}$, $B = \{4,7,8\}$, 求 $\bar{A}$, $\bar{B}$, $\bar{A} \cap \bar{B}$.

解：　$\bar{A} = \{1,2,6,7,8\}$　$\bar{B} = \{1,2,3,5,6\}$

$\bar{A} \cap \bar{B} = \{1,2,6\}$　$\bar{A} \cup \bar{B} = \{1,2,3,5,6,7,8\}$

Example 2.157　Let $I = \{1,2,3,4,5,6,7,8\}$, $A = \{3,4,5\}$, $B = \{4,7,8\}$, find $\bar{A}$, $\bar{B}$, $\bar{A} \cap \bar{B}$.

Solution　$\bar{A} = \{1,2,6,7,8\}$,　$\bar{B} = \{1,2,3,5,6\}$,

$\bar{A} \cap \bar{B} = \{1,2,6\}$,　$\bar{A} \cup \bar{B} = \{1.2,3,5,6,7,8\}$.

第三章　初等代数公式的表达
(Elementary Algebraic Expressions)

§3.1　有理数的运算(Operation of Rational Numbers)

一、加法(Addition)

(一) 加法术语(Terms of Addition)

1. 和(Sums)

加法的解称为和.

The sum is the answer in an addition.

2. 加数(Addends)

被加的数称为加数.

Addends are the numbers that are added.

3. 加法的符号(Signs of Addition)

加法的符号为“+”.

A plus “+” sign indicates addition.

4. 加法的解(Solution of Addition)

加数 + 加数(被加数) = 和(合计)

The terms of addition are:

Addend + Addend (Augend) = Sum (Total)

例 3.1　求下列数的和

$$984+1,292+2,484+337+4,069$$

解法：

```
   984
 1,292
 2,484 ← 加数
   337
+) 4,069
 9,166 ← 和
```

验算：

```
   26      个位数相加
   34      十位数相加
   18      百位数相加
+)  7      千位数相加
 9,166
```

答：9,166

Example 3.1　Find the sum of the following figures:

$$984+1,292+2,484+337+4,069$$

Solution:

```
   984
 1,292
 2,484 ← addends
   337
+) 4,069
 9,166 ← sum
```

Check:

```
   26    sum of units' column
   34    sum of tens' column
   18    sum of hundreds' column
+)  7    sum of thousands' column
 9,166
```

Answer: 9,166

(二) 加法定理(Additive Theorems)

1. 同号两数相加,取相同的符号,并把绝对值相加.

When adding two real numbers with the same signs, keep the same and add their absolute values.

例 3.2(Example 3.2)　$5+3=8$;　$(-5)+(-3)=-8$

英文读作：Five plus three equals eight. Minus five plus minus three equals minus eight.

2. 绝对值不相等的异号两数相加，取绝对值较大的加数的符号，并用值大的数减去值小的数，绝对值相等的异号两数相加得 0.

When adding two real numbers with the opposite signs of the absolute values, get the sign of the addend of the greater absolute value, and use the big number of absolute value to subtract the small number of absolute value. The two numbers with the opposite signs of the same absolute values plus each other equals zero.

例 3.3(Example 3.3)　(1) $5+(-3)=2$

(2) $3+(-5)=-2$

(3) $5+(-5)=0$

英文读作：(1) Five plus minus 3 equals two;

(2) Three plus minus five equals minus two;

(3) Five plus minus five equals zero.

3. 一个数同 0 相加，仍得这个数.

A number plus zero is still the number.

例 3.4(Example 3.4)　$(-5)+0=-5$

英文读作：Minus five plus zero equals minus five.

二、减法(Subtraction)

值相等但符号相异的两个数称为相反数. 如 6 的相反数为 -6，a 的相反数为 $-a$.

The two numbers with the same values but the opposite signs are called additive inverses. For instance, -6 is the additive inverse of 6 and $-a$ is the additive inverse of a.

(一) 减法术语(Terms of Subtraction)

1. 被减数(Minuends)

从中减去的数叫被减数.

The minuend is the number from which you subtract.

2. 减数(Subtrahends)

减去的数称为减数.

The subtrahend is the number you subtract.

3. 差(Remainders or Differences)

差是减法的解.

The remainder or difference is the answer in subtraction.

4. 减号(Signs of Subtraction)

减号用符号“-”表示.

The minus sign “-” indicates subtraction.

5. 减法的求解(Solution of Subtraction)

被减数 - 减数 = 差

Minuend - Subtrahend = Remainder

例 3.5 求下列数的差.

7,485 - 2,648

解法:

7,485←被减数

-)2,648←减数

4,837←差

验算:

2,648←减数

+)4,837←差

7,485←被减数

答: 4,837

Example 3.5 Subtract in the following example.

7,485 – 2,648

Solution:

7,485 ← minuend

–)2,648 ← subtrahend

4,837 ← remainder or difference

Check:

2,648 ← subtrahend

+)4,837 ← difference

7,485 ← minuend

Answer: 4,837

(二) 减法法则(Law of Subtraction)

减去一个数,等于加上这个数的相反数,即:

$$a - b = a + (-b)$$

Subtracting a number equals adding the inverse of the number, namely

$$a - b = a + (-b)$$

例 3.6(Example 3.6)

$15 - 5 = 15 + (-5) = 10$

$5a - 6a = 5a + (-6a) = a$

$(-3) - (-5) = (-3) + 5 = 2$

$0 - 7 = 0 + (-7) = -7$

$7.2 - (-4.8) = 7.2 + 4.8 = 12$

三、乘法(Multiplication)

(一) 乘法术语(Terms of Multiplication)

1. 被乘数(Multiplicands)

所乘的数称为被乘数.

The multiplicand is the number you multiply.

2. 乘数(Multipliers)

被乘的数称为乘数.

The multiplier is the number by which you multiply.

3. 积(Products)

乘法的解称为积.

The product is the solution for multiplication.

4. 部分积(Partial Products)

乘数中的任何一位数乘以被乘数所得的积均称为部分积.

A partial product is the product of the multiplicand multiplied by any digit of the multiplier.

5. 因数(Factors)

求积中的任何数都称为因数.

A factor is one of the terms in getting a product in mutiplication.

6. 乘法符号(Sign of Multiplication)

乘法的符号用“×”代表.

The symbol “×” indicates multiplication.

7. 乘法的解(Solution of Multiplication)

被乘数 × 乘数 = 积

Multiplicand × Multiplier = Product

例 3.7　求 36×45 的积.

解：

36←被乘数
× 45←乘数
180←部分积
144 ←部分积
1,620 ←积

验算：

45←乘数
× 36←被乘数
270←部分积
135 ←部分积
1,620←积

答：1,620

Example 3.7

Multiply 36 by 45.

Solution:

36←multiplicand
× 45←multiplier
180←partial product
144 ←partial product
1,620 ←product

Check:

45←multiplier
× 36←multiplicand
270←partial product
135 ←partial product
1,620←product

Answer: 1,620

(二) 乘法定律(Multiplication Theorems)

1. 两数相乘,同号得正,异号得负,绝对值相乘.

$$ab = c;\quad (-a)(-b) = c;\quad (-a)b = -c$$

In multiplication, if a and b have the same sign, then a times b is positive, and if a and b have the opposite signs, then a times b is negative. Let their absolute values be multiplied by each other.

例 3.8(Example 3.8)　$2\times 3=6$;　　$(-2)\times(-3)=6$

$(-2)\times 3=-6$;　　$2\times(-3)=-6$

2. 任何数同零相乘,都得零;两数相乘得零,其中必有一个数为零;两数相乘不为零,则两个数均不等于零.例:

$a \cdot 0 = 0$;

$ab = 0$ ($a = 0$ 或 $b = 0$);

$ab \neq 0$ ($a \neq 0, b \neq 0$)

For any number times zero, the answer is zero. If $ab = 0$, then either $a = 0$ or $b = 0$. If both $a \neq 0$ and $b \neq 0$, then $ab \neq 0$, that is, tha product of two nonzero real numbers is always nonzero.

例 3.9(Example 3.9)

$(-6) \times 0 = 0$; $0 \times 5 = 0$; $4 \times (-2) = -8$

四、除法(Division)

(一) 除法术语(Terms of Division)

1. 被除数(Dividends)

所除的数为被除数.

The dividend is the number you divide.

2. 除数(Divisors)

被除的数称为除数.

The divisor is tha number by which you divids.

3. 商(Quotients)

除法所得的解称为商.

The quotient is the answer in division.

4. 余数(Remainders)

除不开所剩余的数为余数.

When the division is not exact, the number left is the remainder.

5．部分被除数(Partial Dividends)

添加到余数的被除数部分称为部分被除数．

The partial dividend is part of the dividend annexed to the remainder.

6．除号(Sign of Division)

除号用“÷”或“—”代表．

The symbol “÷”or“—” indicates division.

例3.10 求42,574除以7的商

解：

6,082 ← 商

除数 → 7 ⟌ 42,574 ← 被除数

验算：

6,082 ← 商

×) 7 ← 除数

42,574 ← 被除数

答：商为6,082.

Example 3.10

Divide 42,574 by 7.

Solution:

6,082 ← quotient

divisor → 7 ⟌ 42,574 ← dividend

Check:

6,082 ← quotient

×) 7 ← divisor

42,574 ← dividend

Answer: The quotient is 6,082.

除以一个数，等于乘上这个数的倒数，0不能做除数，即

To divide a number means to multiply the inverse or reciprocal of the

number, and division by zero is not permissible, namely

$$a \div b = a(\frac{1}{b}) = ab^{-1}; \quad b \neq 0$$

例 3.11(Example 3.11) $12 \times 3 = 12 \times \frac{1}{3} = 4$

(二) 除法定义(Definitions of Division)

1. 两数相除,同号得正,异号得负,并把绝对值相除.

$$a \div b = c; \quad (-a) \div (-b) = c; \quad (-a) \div b = -c$$

In division, if a and b have the same signs, then a/b is positive, and if a and b have opposite signs, then a/b is negative. Let their absolute values be divided by each other.

例 3.12(Example 3.12) $(-12) \div (-3) = 4$; $(-15) \div 3 = -5$

2. 零除以任何一个不等于零的数,都得零.

Zero divided by any real number which is not zero equals zero.

$$0 \div a = 0; \quad 0 \div b = 0$$

例 3.13(Example 3.13) $0 \div 3 = 0$

五、乘方(Power)

(一) 乘方定义(Definitions of Power)

1. 如果 n 为正整数,b 为任一实数,则:

$$b^n = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{n\text{个因子}}$$

数字 b 称为底,n 称为指数.

If n is a positive integer and b any real number, then,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$$

The number b is called the base and n the exponent.

2. 正数的任何次幂都是正数.

Any power of a positive number is positive.

例 3.14(Example 3.14) (1) $(2)^3 = 2 \times 2 \times 2 = 8$

(2) $(2)^4 = 2 \times 2 \times 2 \times 2 = 16$

(3) $(2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$

英文读作: (1) Three cubed equals two times two times two equals eight.

(2) The fourth power of two equals two times two times two times two equals sixteen.

(3) The minus second power of two equals one over the second power of two equals one over four.

3. 负数的奇次幂是负数,负数的偶次幂是正数.

The odd power of a negative number is negative and the even power of a negative number is positive.

例 3.15(Example 3.15)

(1) $(-2)^3 = (-2) \times (-2) \times (-2) = -8$

(2) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

英文读作: (1) The third power of minus two equals minus two times minus two times minus two equals minus eight.

(2) Minus two to the fourth power equals minus two times minus two times minus two times minus two equals sixteen.

（二）乘方法则（Rules of Power）

1. $b^m b^n = b^{m+n} = \underbrace{b \cdot b \cdot \cdots \cdot b}_{m\text{个因子(factors)}} \cdot \underbrace{b \cdot \cdots \cdot b}_{n\text{个因子(factors)}}$

因为式子右边因子 b 的总数为 $m + n$，所以该式等于 b^{m+n}.

英文读作：b to the mth power times b to the nth power equals b to the power of m plus n. Since the total number of factors b on the right is m plus n, this expression is equal to b to the power of m plus n.

例 3.16（Example 3.16）

(1) $2^3 \cdot 2^4 = 2^{3+4} = 2^7$　　(2) $x^3 \cdot x^4 = x^7$

2. 如果（If）$a \neq 0$,

$$\frac{b^m}{b^n} = \frac{b^n b^{m-n}}{b^n} = \frac{b^n}{b^n} b^{m-n}$$

$$= 1 \cdot b^{m-n} = \begin{cases} b^{m-n} & \text{如果(if) } m > n \\ 1 & \text{如果(if) } m = n \\ \dfrac{1}{b^{n-m}} & \text{如果(if) } m < n \end{cases}$$

英文读作：Let a not be equal to zero. If m is greater than n, then b to the mth power over b to the nth power equals b to the power of m minus n; if m equals n, then b to the mth power over b to the nth power equals one; if m is less than n, then b to the mth power over b to the nth power equals one over b to the power of n minus m.

例 3.17（Example 3.17）

(1) $\dfrac{2^5}{2^2} = 2^{5-2} = 2^3$　　(2) $\dfrac{2^2}{2^2} = 1$　　(3) $\dfrac{2^2}{2^5} = \dfrac{1}{2^{5-2}} = \dfrac{1}{2^3}$

3. $(b^m)^n = b^{mn} = \underbrace{b^m \cdot b^m \cdot \cdots \cdot b^m}_{n\text{个}b^m\,(n\text{ factors }b^m)}$

英文读作：The nth power of b to the mth power equals the m times n power of b . Since b to the mth power equals b times b times etc., times b, where b occurs as a factor m times, and since the number of such groups of m factors is n, the total number of factors is m times n .

例 3.18(Example 3.18)

(1) $(2^3)^2 = 2^{3\cdot 2} = 2^6$ (2) $(x^3)^2 = x^{3\cdot 2} = x^6$

4. $(ab)^m = a^m b^m$

英文读作：The mth power of the quantity a times b equals a to the m th power times b to the mth power.

例 3.19(Example 3.19)

(1) $(2\cdot 3)^5 = 2^5 \cdot 3^5$ (2) $(xy)^5 = x^5 y^5$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$

英文读作：The mth power of a over b equals a to the mth power over b to the mth power, if only b is not equal to zero.

例 3.20(Example 3.20)

(1) $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$ (2) $\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$

六、混合运算次序(Order of Mixed Operation)

在进行代数式简化运算时,一般次序为:

① 先进行括号内的运算;

② 其次乘方运算;

③ 再从左到右做乘除运算;

④ 最后由左至右做加减.

When simplifying an algebraic expression, the general order is

① Carry out all operations within parentheses;

② Do exponents next;

③ Complete multiplications and divisions, working from left to right;

④ Finally, do additions and subtractions, working from left to right.

例 3.21(Example 3.21) 计算(Calculate)

$$2\frac{1}{5}\times(\frac{1}{3}-\frac{1}{2})\times\frac{3}{11}\div 1\frac{1}{4}$$

解(Solution):

$$2\frac{1}{5}\times(\frac{1}{3}-\frac{1}{2})\times\frac{3}{11}\div 1\frac{1}{4}$$

$$=\frac{11}{5}\times(-\frac{1}{6})\times\frac{3}{11}\div\frac{5}{4}$$

$$=-\frac{11}{30}\times\frac{3}{11}\div\frac{5}{4}$$

$$=-\frac{1}{10}\div\frac{5}{4}=-\frac{1}{10}\times\frac{4}{5}=-\frac{4}{50}=-\frac{2}{25}$$

例 3.22(Example 3.22) 计算(Calculate)

$$-10+8\div(-2)^2-(-4)\times(-3)$$

解(Solution):

$$-10+8\div(-2)^2-(-4)\times(-3)$$

$$= -10 + 8 \div 4 - (-4) \times (-3)$$
$$= -10 + 2 - 12 = -20$$

七、运算律(Properties of Operation)

(一) 交换律(Commutative Properties)

$$a + b = b + a; \qquad a \cdot b = b \cdot a$$

英文读作：a plus b equals b plus a；a times b equals b times a .

两数相加或相乘，交换其位置，它们的值不变.

The commutative properties say that the order in which we either add or multiply real numbers doesn't matter.

例 3.23(Example 3.23)

计算(Calculate) $30 + (-20)$； $(-20) + 30$

解(Solution)： $30 + (-20) = 30 - 20 = 10$

$(-20) + 30 = -20 + 30 = 10$

即(Namely)： $30 + (-20) = (-20) + 30$

英文读作：Thirty plus minus twenty equals minus twenty plus thirty.

例 3.24(Example 3.24)

计算(Calculate) $5 \times (-6)$； $(-6) \times 5$

解(Solution)： $5 \times (-6) = -30$； $(-6) \times 5 = -30$

即(Namely)： $5 \times (-6) = (-6) \times 5$

英文读作：Five times minus six equals minus six times five.

(二) 结合律(Associative Properties)

$$a + (b + c) = (a + b) + c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

英文读作：a plus the quantity b plus c equals the quantity a plus b plus c .

a times open parenthesis b times c close parenthesis equals open parenthesis a times b close parenthesis times c .

三个数相加(或相乘)，先将后两个数相加(或相乘)，再加(或乘)前一个数，或先把前两个数相加(或相乘)再和第三个数相加(或乘)，其值不变.

The associative properties tell us that the way real numbers are grouped when they are either addad or multiplied doesn't matter.

例 3.25(Example 3.25)

计算(Calculate) $[8+(-5)]+(-4)$

$8+[(-5)+(-4)]$

解(Solution)：$[8+(-5)]+(-4)=3+(-4)=-1$

$8+[(-5)+(-4)]=8+(-9)=-1$

即(Namely)：$[8+(-5)]+(-4)=8+[(-5)+(-4)]$

例 3.26(Example 3.26)

计算(Calculate) $[3\times(-4)]\times(-5)$； $3\times[(-4)\times(-5)]$

解(Solution)：$[3\times(-4)]\times(-5)=(-12)\times(-5)=60$

$3\times[(-4)\times(-5)]=3\times 20=60$

即(Namely)：$[3\times(-4)]\times(-5)=3\times[(-4)\times(-5)]$

(三) 分配律(Distributive Properties)

$$a\cdot(b+c)=a\cdot b+a\cdot c$$

$$(b+c)\cdot a=b\cdot a+c\cdot a$$

英文读作：a times the quantity b plus c is equal to a times b plus a times c .

The quantity b plus c times a is equal to b times a plus c times a .

两数之和与一个数相乘,等于把两个数分别与这个数相乘,再把两个积相加.

The product of a real number times the sum of two others is the same as the sum of the products of the first number times each of the others.

例 3.27(Example 3.27)

计算(Calculate) $5\times[3+(-7)]$; $5\times3+5\times(-7)$

解(Solution): $5\times[3+(-7)]=5\times(-4)=-20$

$5\times3+5\times(-7)=15-35=-20$

即(Namely): $5\times[3+(-7)]=5\times3+5\times(-7)$

例 3.28(Example 3.28)

计算(Calculate) $(\frac{1}{4}+\frac{1}{6}-\frac{1}{2})\times12$

解(Solution): $(\frac{1}{4}+\frac{1}{6}-\frac{1}{2})\times12$

$=\frac{1}{4}\times12+\frac{1}{6}\times12-\frac{1}{2}\times12=3+2-6=-1$

§3.2 整式的运算(Operation of Integral Expressions)

一、几个基本定义(Several Basic Definitions)

(一) 代数式(Algebraic Expressions)

在加、减、乘、除、乘方和开方等运算中由数字和字母(或变量)

表达的式子称为代数式.

In the operation of addition, subtraction, multiplication, division, exponentiation (raising to powers) and extraction of roots, the expression formed from any combination of numbers and variables is called an algebraic expression.

例 3.29(Example 3.29)

7，x，$2x-3y+1$，$\frac{5x^3-1}{4xy+1}$，πr^2 和 $\pi r\sqrt{r^2+h^2}$ 均为代数式.

7，x，$2x-3y+1$，$\frac{5x^3-1}{4xy+1}$，πr^2 and $\pi r\sqrt{r^2+h^2}$ are all algebraic expressions.

(二) 整式(Integral Expressions)

分母中不含有字母(或变量)的代数式叫整式. 整式中包括单项式和多项式.

The algebraic expression without letters or variables in denominators is called an integral expression. An integral expression includes monomials and polynomials.

例 3.30(Example 3.30)

$2\pi(a-5)$，$2x+3xy^2+1$，$5a-3a^2b+b^2a-1$ 和 $-\frac{3}{5}xy^2$ 均为整式.

$2\pi(a-5)$，$2x+3xy^2+1$，$5a-3a^2b+b^2a-1$ and $-\frac{3}{5}xy^2$ are all integral expressions.

(三) 单项式(Monomials)

1. 定义(Definition)

只有一个数字与字母(或变量)的积的代数式叫做单项式.

The algebraic expression containing only one product of numbers and letters or variables is called a monomial.

例 3.31(Example 3.31)

$4x$, $-7xy^2$, $\frac{1}{3}a^2b^2$, 5, ab 和 $-n$ 均为单项式.

$4x$, $-7xy^2$, $\frac{1}{3}a^2b^2$, 5, ab and $-n$ are all monomials.

2. 单项式的系数与次数(The coefficient and degree of a monomial)

在单项式中,数字叫做单项式的系数,所有字母(或变量)指数之和叫做单项式的次数(不标指数的变量被看作是一次幂).

A number is called the coefficient of a monomial. The sum of all the exponents of letters or variables is called the degree of the monomial(the variable marked uo exponent is regarded as a first power).

例 3.32(Example 3.32)

(1) $4x$ 系数是 4 的一次单项式.

The coefficient of x is 4 and the monomial has degree 1.

(2) $-7xy^2$ 系数为 -7 的三次单项式.

The coefficient of xy^2 is -7 and the monomial has degree 3.

(3) $\frac{1}{3}a^2b^2$ 系数为$\frac{1}{3}$ 的四次单项式.

The coefficient of a^2b^2 is $\frac{1}{3}$ and the monomial has degree 4.

(4) 5 为常数的 0 次单项式.

5 is a constant and has degree 0.

(5) ab 系数为 1 的二次单项式.

The coefficient of ab is 1 and the monomial has degree 2.

(6) $-n$ 系数为 -1 的一次单项式

The coefficient of n is -1 and the monomial has degree 1.

(四) 多项式(Polynomials)

1. 定义(Definition)

有两项或两项以上单项式的代数和叫做多项式.

The algebraic sum with two or more terms is called a multinomial or polynomial.

2. 多项式的项(Terms of a Polynomial)

在多项式中,每个单项式叫做多项式的项,不含变量的项叫做常数项.

In a polynomial, each monomial is called the terms of the polynomial. The term which contains no variables is called a constant term of the polynomial.

例 3.33(Example 3.33)

(1) 在多项式 $4x-5$ 中,$4x$, -5 是它的项,其中 -5 是常数项.

In the polynomial of $4x-5$, $4x$ and -5 are its terms in which -5 is a constant term.

(2) 在多项式 $6x^2-2x+7$ 中,$6x^2$, $-2x$ 和 7 是它的项,其中 7 是常数项.

In $6x^2-2x+7$, $6x^2$, $-2x$ and 7 are its terms in which 7 is a constant term.

(3) 在多项式 a^3+ab+b^2 中,a^3, ab, b^2 是它的项,没有常数项.

In a^3+ab+b^2, a^3, ab and b^2 are its terms in which there is no constant term.

3. 多项式的次数(Degrees of Polynomials)

在多项式中,系数不为零的所有项中最高的次数叫做多项式

的次数.

The highest degree of all terms that appear with nonzero coefficients in a polynomial is called the degree of the polynomial.

例 3.34(Example 3.34)

(1) $4x - 5$　一次二项式(a binomial with degree 1).

(2) $6x^2 - 2x + 7$　二次三项式(a trinomial with degree 2).

(3) $a^3 + ab + b^2$　三次三项式(a trinomial with degree 3).

二、整式的加减(Addition and Subtraction of Integral Expressions)

(一) 合并同类项(Combining Similar Terms)

1. 同类项(Similar Terms)

除数字系数外其它均相同的项称为同类项.

The terms that are identical, except for their numerical coefficients, are called similar terms or like terms.

例 3.35(Example 3.35)

$$4x^2 + 2y - 3xy + 7 + 3y - 8x^2 - 2$$

在这个多项式中,$4x^2$ 与 $-8x^2$ 是同类项,$2y$ 与 $3y$ 是同类项,7 与 -2 是同类项,但 $-3xy$ 没有同类项.

In the polynomial, $4x^2$ and $-8x^2$ are like terms, $2y$ and $3y$ are like terms, 7 and -2 are like terms, but $-3xy$ has no like term.

2. 合并同类项(Combination of Similar Terms)

将多项式中的同类项合并成一项.

The way to combine the similar terms of a polynomial as one is called the combination of similar terms.

3. 合并法则(Properties of Combination)

同类项的系数相加,所得的结果作为系数,字母(或变量)和字母(或变量)的指数不变.

The coefficients of similar terms are grouped together and the result is used as the coefficient and the letters or variables and their exponents are not changed.

例 3.36 将多项式 $3x^3 + x^3$ 合并同类项.

解: $3x^3 + x^3 = (3 + 1)x^3 = 4x^3$

Example 3.36 Combine the similar terms of the polynomial $3x^3 + x^3$.

Solution $3x^3 + x^3 = (3 + 1)x^3 = 4x^3$

例 3.37 将下列多项式合并同类项

$$4x^2 - 8x + 5 - 3x^2 + 6x - 2$$

解: $4x^2 - 8x + 5 - 3x^2 + 6x - 2$

$= (4x^2 - 3x^2) + (-8x + 6x) + (5 - 2)$

$= x^2 - 2x + 3$

Example 3.37 Combine the similar terms of the following polynomial

$$4x^2 - 8x + 5 - 3x^2 + 6x - 2$$

Solution $4x^2 - 8x + 5 - 3x^2 + 6x - 2$

$= (4x^2 - 3x^2) + (-8x + 6x) + (5 - 2)$

$= x^2 - 2x + 3$

(二) 去括号与添括号(Removing and Adding Brackets)

1. 去括号(Removing Brackets)

括号前是"+",把括号和它前面的"+"去掉,括号里的各项都不变符号;括号前是"-",把括号和它前面的"-"去掉,括号里各项都改变符号.

If the sign before a bracket is "+", then remove the brackets and the sign "+" without changing any signs of the terms in brackets; if the sign before a bracket is "−", then remove the brackets and the sign "−", and change the signs of all the terms in brackets.

例 3.38　将下列多项式去括号

(1) $a+(-b+c-d)$　　(2) $a-(-b+c-d)$

解：(1) $a+(-b+c-d)=a-b+c-d$

(2) $a-(-b+c-d)=a+b-c+d$

Example 3.38　Remove the brackets of the following polynomials

(1) $a+(-b+c-d)$　　(2) $a-(-b+c-d)$

Solution　(1) $a+(-b+c-d)=a-b+c-d$

(2) $a-(-b+c-d)=a+b-c+d$

例 3.39　将下列多项式先去括号，再合并同类项

(1) $8a+2b+(5a-b)$　　(2) $6a-2(a-c)$

解：(1) $8a+2b+(5a-b)$

$=8a+2b+5a-b=13a+b$

(2) $6a-2(a-c)=6a-2a+2c=4a+2c$

Example 3.39　Remove the brackets first and then combine the similar terms of the following polynomials

(1) $8a+2b+(5a-b)$　　(2) $6a-2(a-c)$

Solution　(1) $8a+2b+(5a-b)$

$=8a+2b+5a-b=13a+b$

(2) $6a-2(a-c)=6a-2a+2c=4a+2c$

2. 添括号(Adding Brackets)

添括号后，括号前面是"+"，括到括号里的各项都不变符号；添括号后，括号前面是"−"，括到括号里的各项都改变符号.

After adding brackets, all the terms are not changed in signs if the sign before the bracket is positive. If the sign before the bracket is negative, then all the terms must be changed in signs after adding brackets.

例 3.40(Example 3.40)

$$x^3 - 5x^2 - 4x + 9 = x^3 - 5x^2 + (-4x + 9)$$
$$x^3 - 5x^2 - 4x + 9 = x^3 - 5x^2 - (4x - 9)$$

(三) 整式的加减步骤(Procedure of Addition and Subtraction)

整式加减的一般步骤(General Procedures):

① 如果遇到括号,按去括号法则先去括号;

For the expressions with brackets, first remove the brackets;

② 合并同类项.

Combine similar terms.

例 3.41(Example 3.41)

$$5x^2y + (-2x^2y) + 2xy^2 + (-4x^2y)$$
$$= 5x^2y - 2x^2y + 2xy^2 - 4x^2y = -x^2y + 2xy^2$$

例 3.42(Example 3.42)

$$(3x^2 - 6x + 5) + (4x^2 + 7x - 6)$$
$$= (3x^2 - 6x + 5) + 4x^2 + 7x - 6$$
$$= 7x^2 + x - 1$$

例 3.43(Example 3.43)

$$(2x^2 + xy + 3y^2) - (x^2 - xy + 2y^2)$$
$$= 2x^2 + xy + 3y^2 - x^2 + xy - 2y^2$$
$$= x^2 + 2xy + y^2$$

三、整式的乘法(Multiplication of Integral Expressions)

(一) 同底数幂相乘法则

(Properties of Multiplication for Powers with the Same Base)

同底数幂相乘,底数不变,指数相加.

$$a^m \cdot a^n = a^{m+n} \quad (m, n \text{ 都是正整数})$$

$$a^m \cdot a^n \cdot a^p = a^{m+n+p} \quad (m, n, p \text{ 都是正整数})$$

In multiplication of powers with the same base, keep the base and add their exponents.

$$a^m \cdot a^n = a^{m+n} \quad (m, n \text{ are positive integers})$$

$$a^m \cdot a^n \cdot a^p = a^{m+n+p} \quad (m, n, p \text{ are positive integers})$$

例 3.44(Example 3.44) 计算(Calculate)

(1) $10^7 \times 10^4$ (2) $x^2 \cdot x^5$

解(Solution): (1) $10^7 \times 10^4 = 10^{7+4} = 10^{11}$

(2) $x^2 \cdot x^5 = x^{2+5} = x^7$

例 3.45(Example 3.45) 计算(Calculate)

(1) $2^3 \times 2^4 \times 2^5$ (2) $y \cdot y^2 \cdot y^3$

解(Solution): (1) $2^3 \times 2^4 \times 2^5 = 2^{3+4+5} = 2^{12}$

(2) $y \cdot y^2 \cdot y^3 = y^{1+2+3} = y^6$

(二) 幂的乘方与积的乘方

(Powers of Exponents and Products)

1. 幂的乘方法则(Property of Power of Exponents)

底数不变,指数相乘.

$$(a^m)^n = a^{mn} \quad (m, n \text{ 都是正整数}).$$

The base remains the same and their exponents are multiplied by

each other.

$(a^m)^n = a^{mn}$ (m and n are both positive integers).

例 3.46(Example 3.46) 计算(Calculate)

(1) $(10^7)^2$ (2) $(z^4)^4$

(3) $-(y^4)^3$ (4) $(a^m)^4$

解(Solution):

(1) $(10^7)^2 = 10^{7\times2} = 10^{14}$

(2) $(z^4)^4 = z^{4\times4} = z^{16}$

(3) $-(y^4)^3 = -y^{4\times3} = -y^{12}$

(4) $(a^m)^4 = a^{m\times4} = a^{4m}$

2. 积的乘方法则(Property of Power of Product)

把积的每一个因式分别乘方,再把所得的幂乘方,即

$(ab)^n = a^nb^n$ (n 为正整数)

$(abc)^n = a^nb^nc^n$ (n 为正整数)

Multiply each factor of a product by its exponent and then multiply the resulting powers.

$(ab)^n = a^nb^n$ (n is a positive integer)

$(abc)^n = a^nb^nc^n$ (n is a positive integer)

例 3.47(Example 3.47)

计算(Calculate) (1) $(-3x)^3$ (2) $(-5ab)^2$

(3) $(xy^2)^2$ (4) $(-2xy^3z^2)^4$

解(Solution):

(1) $(-3x)^3 = (-3)^3x^3 = -27x^3$

(2) $(-5ab)^2 = (-5)^2a^2b^2 = 25a^2b^2$

(3) $(xy^2)^2 = x^2\cdot(y^2)^2 = x^2y^4$

(4) $(-2xy^3z^2)^4 = (-2)^4(x)^4(y^3)^4(z^2)^4$

$$16x^4y^{12}z^8$$

例 3.48(Example 3.48)

计算(Calculate) (1) $a^3 \cdot a^4 \cdot a + (a^2)^4 + (-2a^4)^2$

(2) $2(x^3)^2 \cdot x^3 - (3x^3)^3 + (5x)^2 \cdot x^7$

解(Solution):

(1) $a^3 \cdot a^4 \cdot a + (a^2)^4 + (-2a^4)^2$

$= a^{3+4+1} + a^{2\times4} + (-2)^2 \cdot (a^4)^2$

$= a^8 + a^8 + 4a^8 = 6a^8$

(2) $2(x^3)^2 \cdot x^3 - (3x^3)^3 + (5x)^2 \cdot x^7$

$= 2x^6 \cdot x^3 - 27x^9 + 25x^2 \cdot x^7$

$= 2x^9 - 27x^9 + 25x^9 = 0$

(三) 单项式的乘法(Multiplication of Monomials)

单项式相乘,把它们的系数、相同字母(或变量)分别相乘.

For the multiplication of monomials, multiply their coefficients and the same letters or variables, respectively.

例 3.49(Example 3.49)

$(-5a^2b^3) \cdot (-3a)$

$= [(-5) \cdot (-3)](a^2 \cdot a)b^3 = 15a^3b^3$

例 3.50(Example 3.50)

$(4 \times 10^5)(5 \times 10^6)(3 \times 10^4)$

$= (4 \times 5 \times 3)(10^5 \times 10^6 \times 10^4)$

$= 60 \times 10^{15} = 6 \times 10^{16}$

例 3.51(Example 3.51)

$$\frac{2}{3}x^3y^2 \cdot (-\frac{3}{2}xy^2)^2$$
$$= \frac{2}{3}x^3y^2 \cdot \frac{9}{4}x^2y^4$$
$$= (\frac{2}{3} \times \frac{9}{4})(x^3 \cdot x^2)(y^2 \cdot y^4)$$
$$= \frac{3}{2}x^5y^6$$

(四) 单项式与多项式相乘(Monomials Times Polynomials)

用单项式去乘多项式的每一项,再把所得的积相加.

Have a monomial multiplied by each term of a polynomial and then add their products.

例 3.52(Example 3.52)

$$(-4x) \cdot (2x^2 + 3x - 1)$$
$$= (-4x) \cdot (2x^2) + (-4x) \cdot 3x + (-4x) \cdot (-1)$$
$$= -8x^3 - 12x^2 + 4x$$

例 3.53(Example 3.53)

$$-2a^2 \cdot (\frac{1}{2}ab + b^2) - 5a(a^2b - ab^2)$$
$$= -a^3b - 2a^2b^2 - 5a^3b + 5a^2b^2$$
$$= -6a^3b + 3a^2b^2$$

(五) 多项式的乘法(Multiplication of Polynomials)

先用一个多项式的每一项乘以另一个多项式的每一项,再把所得的积相加.

Have each term of one polynomial multiplied by each term of another and then add the resulting products.

例 3.54(Example 3.54)

$$(x + 2y)(5a + 3b)$$
$$= x \cdot 5a + x \cdot 3b + 2y \cdot 5a + 2y \cdot 3b$$
$$= 5ax + 3bx + 10ay + 6by$$

例 3.55(Example 3.55)

$$(x + y)(x^2 - xy + y^2)$$
$$= x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3 = x^3 + y^3$$

例 3.56(Example 3.56)

$$(x + 30)(x - 40)$$
$$= x^2 - 40x + 30x - 1,200 = x^2 - 10x - 1,200$$

四、整式的除法(Division of Integral Expressions)

(一) 同底数幂的除法(Division of Powers with the Same Base)

1. 同底数幂相除,底数不变,指数相减,即

$$a^m \div a^n = a^{m-n}$$

其中,$a \neq 0$, m,n 都是正整数,且 $m > n$.

In division of powers with the same base, keep the base and subtract their exponents.

$$a^m \div a^n = a^{m-n}$$

where $a \neq 0$, m and n are both positive integers and $m > n$.

例 3.57(Example 3.57)

$$x^8 \div x^2 = x^{8-2} = x^6$$
$$(ab)^5 \div (ab)^2 = (ab)^{(5-2)} = (ab)^3 = a^3b^3$$
$$y^{n+2} \div y^2 = y^{n+2-2} = y^n$$

2. 任何不等于零的数的零次幂都等于 1.

The 0th power of any nonzero real number equals 1.

$$a^0 = 1 \qquad (a \neq 0)$$

例 3.58(Example 3.58)

$$3^0 = 1 \qquad 10^0 = 1 \qquad (abc)^0 = 1 \qquad (xy^5)^0 = 1$$

3. 任何不等于零的数的 $-p$（p 是正整数）次幂，等于这个数的 p 次幂的倒数.

The minus pth power of any nonzero real number equals the reciprocal of the number to the pth power. Here p is a positive integer, that is

$$a^{-p} = \frac{1}{a^p}.$$

例 3.59(Example 3.59)

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1,000}$$

$$5^0 \times 10^{-2} = 1 \times \frac{1}{10^2} = \frac{1}{100}$$

（二）单项式除以单项式(Monomials Divided by Monomials)

单项式相除，把系数、同底数幂分别相除，作为商的因式.对于只在被除式里含有的字母(或变量)，则连同它的指数作为商的一个因式.

In division of monomials, the coefficients and powers with the same bases are divided, respectively, by each other, which are regarded as the factor of the quotient. For the letter (or variable) involved only in the dividend, we regard it, as well as its exponent, as a factor of the quotient.

例 3.60(Example 3.60)

(1) $28x^4y^2 \div 7x^3y = (28 \div 7) \cdot x^{4-3}y^{2-1} = 4xy$

(2) $-5a^5b^3c \div 15a^4b^3$

$= [(-5) \div 15] \cdot a^{5-4}b^{3-3}c = -\frac{1}{3}ac$

(3) $-a^2x^4y^3 \div (-\frac{5}{6}axy^2) = \frac{6}{5}ax^3y$

(4) $(6x^2y^3)^2 \div (3xy^2)^2 = 36x^4y^6 \div 9x^2y^4 = 4x^2y^2$

(三) 多项式除以单项式(Polynomials Divided by Monomials)

多项式除以单项式,先把这个多项式的每一项除以这个单项式,再把所得的商相加.

When a polynomial is divided by a monomial, first have each term of the polynomial divided by the monomial, then add their quotients.

例 3.61(Example 3.61)

$(28a^3 - 14a^2 + 7a) \div 7a$

$= 28a^3 \div 7a - 14a^2 \div 7a + 7a \div 7a$

$= 4a^2 - 2a + 1$

例 3.62(Example 3.62)

$(36x^4y^3 - 24x^3y^2 + 3x^2y^2) \div (-6x^2y)$

$= -6x^2y^2 + 4xy - \frac{1}{2}y$

例 3.63(Example 3.63)

$[(2x + y)^2 - y(y + 4x) - 8x] \div 2x$

$= (4x^2 + 4xy + y^2 - y^2 - 4xy - 8x) \div 2x$

$= (4x^2 - 8x) \div 2x = 2x - 4$

五、乘法公式及因式分解(Product Formulas and Factoring)

(一) 乘法公式(Product Formulas)

1. 平方差公式(Difference of Squares)

两个数的和与这两个数的差的积等于这两个数的平方差,即

$$(a+b)(a-b)=a^2-b^2$$

The sum of two real numbers times the difference of the two real numbers equals the difference of squares of the two numbers, that is

$$(a+b)(a-b)=a^2-b^2$$

例 3.64(Example 3.64)

化简(Simplify) 102×98

解(Solution): $102\times 98=(100+2)(100-2)=100^2-2^2$

$=10,000-4=9,996$

例 3.65(Example 3.65)

$(y+2)(y-2)(y^2+4)$

解(Solution):

$(y+2)(y-2)(y^2+4)$

$=(y^2-4)(y^2+4)=(y^2)^2-4^2=y^4-16$

2. 完全平方公式(Perfect Squares)

两数和(或差)的平方,等于它们的平方和加上(或减去)它们的积的 2 倍,即

完全平方和公式: $(a+b)^2=a^2+2ab+b^2$

完全平方差公式: $(a-b)^2=a^2-2ab+b^2$

The square of the sum or difference of two real numbers equals their squares of sum plus or minus twice their product, that is

sum of perfect squares: $(a+b)^2 = a^2 + 2ab + b^2$
difference of perfect squares: $(a-b)^2 = a^2 - 2ab + b^2$

例 3.66(Example 3.66)

(1) $102^2 = (100+2)^2 = 100^2 + 2\times 100\times 2 + 2^2$
$= 10,000 + 400 + 4 = 10,404$

(2) $199^2 = (200-1)^2 = 200^2 - 2\times 200 + 1^2$
$= 40,000 - 400 + 1 = 3,9601$

(3) $(\frac{x}{2}+5)^2 - (\frac{x}{2}-5)^2$
$= \frac{x^2}{4} + 5x + 25 - (\frac{x^2}{4} - 5x + 25)$
$= \frac{x^2}{4} + 5x + 25 - \frac{x^2}{4} + 5x - 25 = 10x$

3. 完全立方公式(Perfect Cubes)

两数和(或差)乘以它们的平方和与它们的积的差(或和),等于这两个数的立方和(或差),即

完全立方和公式: $(a+b)(a^2 - ab + b^2) = a^3 + b^3$
完全立方差公式: $(a-b)(a^2 + ab + b^2) = a^3 - b^3$

The sum or the difference of two real numbers times the sum of their squares minus or plus their product equals the sum or the difference of the two real numbers cubed, namely

sum of perfect cubes: $(a+b)(a^2 - ab + b^2) = a^3 + b^3$
difference of perfect cubes: $(a-b)(a^2 + ab + b^2) = a^3 - b^3$

例 3.67(Example 3.67)

(1) $(4+a)(16-4a+a^2) = (4+a)(4^2 - 4a + a^2)$
$= 4^3 + a^3 = 64 + a^3$

(2) $(5x - \frac{1}{2}y)(25x^2 + \frac{5}{2}xy + \frac{1}{4}y^2)$

$$= (5x - \frac{1}{2}y)[(5x)^2 + (5x)(\frac{1}{2}y) + (\frac{1}{2}y)^2]$$
$$= (5x)^3 - (\frac{1}{2}y)^3 = 125x^3 - \frac{1}{8}y^3$$

(二) 因式分解(Factoring)

当两个或两个以上的代数式相乘时,每个式子都叫做乘积的一个**因式**.如在下列乘积

$$(x - y)(x + y)(2x^2 - y)x$$

中,因式为 $(x - y)$,$(x + y)$,$(2x^2 - y)$ 和 x. 这种把一个多项式化为 n 个整式的积的形式,叫做把这个多项式因式分解,也叫做把多项式分解因式.

When two or more algebraic expressions are multiplied, each expression is called a factor of the product. For instance, in the product

$$(x - y)(x + y)(2x^2 - y)x,$$

the factors are $(x - y)$, $(x + y)$, $(2x^2 - y)$ and x. The process of changing a polynomial into a product of n integral expressions is called factoring or factoring a polynomial.

1. 提取公因式(Abstracting Common Factors)

把多项式各项都有的因式提到括号外的过程称为提取公因式.

The process of abstracting the common factors of a polynomial out of their parentheses is called abstracting common factors.

例 3.68(Example 3.68)

因式分解下列各式.

Factor the following expressions.

(1) $20x^2y + 8xy$　　(2) $u(v + w) + 7v(v + w)$

解(Solution):

(1) $20x^2y + 8xy = (4xy)(5x) + (4xy)(2) = 4xy(5x + 2)$

(2) $u(v + w) + 7v(v + w) = (u + 7v)(v + w)$

2. 运用乘积公式(Using Product Formulas)

例 3.69(Example 3.69)

(1) $25t^2 - 16s^2$ (2) $27r^2 - 8c^8$ (3) $x^3 + 64y^3$

解(Solution):

(1) $25t^2 - 16s^2 = (5t)^2 - (4s)^2 = (5t - 4s)(5t + 4s)$

(2) $27r^2 - 8c^8 = (3r)^3 - (2c)^3$

$= (3r - 2c)[(3r)^2 + (3r)(2c) + (2c)^2]$

$= (3r - 2c)(9r^2 + 6rc + 4c^2)$

(3) $x^3 + 64y^3 = x^3 + (4y)^3 = (x + 4y)(x^2 - 4xy + 16y^2)$

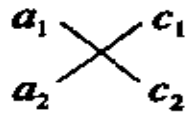
3. 二次三项式 $ax^2 + bx + c$ 的因式分解

(Factoring Trinomial of the Type $ax^2 + bx + c$)

如果 a, b, c 是整数, $a \neq 0$, 有可能将 $ax^2 + bx + c$ 类型的二次三项式因式分解成二项式的乘积. 其中, $a = a_1a_2$, $b = a_1c_2 + a_2c_1$, $c = c_1c_2$. 则有

$$ax^2 + bx + c = a_1a_2x^2 + (a_1c_2 + a_2c_1)x + c_1c_2$$
$$= (a_1x + c_1)(a_2x + c_2)$$

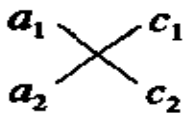
a_1, c_1, a_2, c_2 按如下规律排列



If a, b, and c are integers and $a \neq 0$, it may be possible to factor a trinomial of the type $ax^2 + bx + c$ into a product of two binomials, where if $a = a_1a_2$, $b = a_1c_2 + a_2c_1$ and $c = c_1c_2$, then

$$ax^2 + bx + c = a_1a_2x^2 + (a_1c_2 + a_2c_1)x + c_1c_2$$
$$= (a_1x + c_1)(a_2x + c_2)$$

a_1, c_1, a_2, c_2 can be ordered as follows



按斜线相乘，再相加，所得结果应等于 b，这种分解因式的方法叫做十字相乘法.

Multiplying a_1 by c_2 and a_2 by c_1 along oblique lines and then adding their products, the result is b. this method to factor a trinomial is called cross multiplication.

例 3.70(Example 3.70)

因式分解三项式 $2x^2 + 9x + 4$

Factor the trinomial $2x^2 + 9x + 4$

解： $2x^2 + 9x + 4 = (2x + 1)(x + 4)$

4. 分组分解(Factoring by Grouping)

有时可以用相应方式将一个多项式的项分组，以找出因式分解的方法.

Sometimes we have to group the terms of a polynomial in a certain way in order to see how it can be factored.

例 3.71(Example 3.71)

用分组法分解下列各式.

(Factor the following expressions by grouping)

(1) $ac + d - c - ad$　　(2) $x^2 + 4xy - 9c^2 + 4y^2$

(3) $4x^3 - 8x^2 - x + 2$　　(4) $x^4 + 6x^2y^2 + 25y^4$

解(Solution)：

(1) $ac + d - c - ad = (ac - c) + (d - ad)$

$$= c(a-1) - d(a-1) = (c-d)(a-1)$$

(2) $x^2 + 4xy - 9c^2 + 4y^2 = (x^2 + 4xy + 4y^2) - 9c^2$

$$= (x+2y)^2 - (3c)^2$$

$$= [(x+2y) - 3c][(x+2y) + 3c]$$

$$= (x + 2y - 3c)(x + 2y + 3c)$$

(3) $4x^3 - 8x^2 - x + 2 = 4x^2(x-2) - x + 2$

$$= 4x^2(x-2) - (x-2) = (4x^2 - 1)(x-2)$$

$$= (2x-1)(2x+1)(x-2)$$

(4) $x^4 + 6x^2y^2 + 25y^4 = (x^4 + 10x^2y^2 + 25y^4) - 4x^2y^2$

$$= (x^2 + 5y^2)^2 - (2xy)^2$$

$$= [(x^2 + 5y^2) - 2xy][(x^2 + 5y^2) + 2xy]$$

$$= (x^2 - 2xy + 5y^2)(x^2 + 2xy + 5y^2)$$

§3.3 分式和根式(Fractions and Radicals)

一、分式(Fractions)

(一) 定义(Definition)

两个整式相除,且作为除数的整式中含有字母(或变量),就叫做分式.

Quotients of two algebraic expressions with denominator containing letters (or variables) are called fractional expressions.

例 3.72(Example 3.72) $\dfrac{x^2 - 5x + 1}{x^3 + 7}$, $\dfrac{x^x - 3yz}{5z}$, $\dfrac{1}{4xy}$

以上均为分式.(The above are all fractions)

(二) 基本性质(Basic Property)

分式的分子与分母都乘以(或除以)同一个不等于零的整式,分式的值不变,即

$$\frac{a}{b}=\frac{am}{bm};\quad \frac{a}{b}=\frac{a\div m}{b\div m}\quad (m\neq 0)$$

If the numerator and the denominator of a fractional expression is multiplied or divided by the same nonzero integral expression, then the value of the fraction remains the same, that is

$$\frac{a}{b}=\frac{am}{bm};\quad \frac{a}{b}=\frac{a\div m}{b\div m}\quad (m\neq 0)$$

例 3.73(Example 3.73):

简化(Simplify) $\dfrac{3x^2-5x-2}{x^2-4}$

解(Solution): $\dfrac{3x^2-5x-2}{x^2-4}$

$$=\frac{(3x+1)(x-2)}{(x+2)(x-2)}=\frac{3x+1}{x+2}$$

例 3.74(Example 3.74)

简化(Simplify) $\dfrac{2-x-3x^2}{6x^2-x-2}$

解(Solution):

$$\frac{2-x-3x^2}{6x^2-x-2}=\frac{(1+x)(2-3x)}{(2x+1)(3x-2)}=\frac{-(1+x)}{(2x+1)}$$

或(or) $\dfrac{2-x-3x^2}{6x^2-x-2}=\dfrac{-(3x^2+x-2)}{6x^2-x-2}$

$$=-\frac{(3x-2)(x+1)}{(3x-2)(2x+1)}=-\frac{x+1}{2x+1}$$

(三) 分式运算(Fractional Operations)

1. 分式的乘除法(Multiplication and Division of Fractions)

1) 约分(Cancellation)

把分式的分子与分母分解因式,然后约去分子与分母的公因式的过程叫做约分.

Factor both the numerator and denominator of a fraction and then cancel their common factors. This process is called cancellation.

2) 最简分式(Lowest Terms)

一个分式的分子与分母没有公因式时,叫做最简分式.

A fraction whose numerator and denominator have no common factors is said to be reduced to the lowest or simplest terms.

例 3.75(Example 3.75)

简化(Simplify)

$$(1)\ \frac{-32a^2b^3c}{24b^2cd}\quad (2)\ \frac{m^2-3m}{9-m^2}\quad (3)\ \frac{x^2+4x+3}{x^2+x-6}$$

解(Solution):

$$(1)\ \frac{-32a^2b^3c}{24b^2cd}=-\frac{8b^2c\cdot 4a^2b}{8b^2c\cdot 3d}=-\frac{4a^2b}{3d}$$

$$(2)\ \frac{m^2-3m}{9-m^2}=\frac{m(m-3)}{-(m^2-9)}$$

$$=-\frac{m(m-3)}{(m+3)(m-3)}=-\frac{m}{m+3}$$

$$(3)\ \frac{x^2+4x+3}{x^2+x-6}=\frac{(x+1)(x+3)}{(x-2)(x+3)}=\frac{x+1}{x-2}$$

3) 分式的乘法(Multiplication of Fractions)

分式乘以分式,用分子的积做积的分子,分母的积做积的分

母,即

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

In multiplying a fraction by another fraction, use the product of the numerators as the numerator of the product and the product of the denominators as the denominator of the product, that is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

例 3.76(Example 3.76)

计算(Calculate) $\frac{4x}{3y} \cdot \frac{y}{2x^3}$

解(Solution):

$$\frac{4x}{3y} \cdot \frac{y}{2x^3} = \frac{4x \cdot y}{3y \cdot 2x^3} = \frac{2}{3x^2}$$

例 3.77(Example 3.77)

计算(Calculate) $\frac{a^2-4}{a^2-4a+3} \cdot \frac{a-3}{a^2+3a+2}$

解(Solution):

$$\begin{aligned}
&\frac{a^2-4}{a^2-4a+3} \cdot \frac{a-3}{a^2+3a+2} \\
&= \frac{(a+2)(a-2)}{(a-1)(a-3)} \cdot \frac{a-3}{(a+1)(a+2)} \\
&= \frac{(a+2)(a-2)(a-3)}{(a-1)(a-3)(a+1)(a+2)} \\
&= \frac{a-2}{(a+1)(a-1)} = \frac{a-2}{a^2-1}
\end{aligned}$$

4) 分式的除法(Division of Fractions)

分式除以分式,把除式的分子、分母颠倒位置后,与被除式相乘,即

In dividing a fractional expression by another fractional expression, inverse the numerator and denominator of the divisor, and then have it multiplied by the dividend, that is

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

例 3.78(Example 3.78)

计算(Calculate)

$$\frac{ab^2}{2c^2} \div \frac{-3a^2b^2}{4cd}$$

解(Solution):

$$\frac{ab^2}{2c^2} \div \frac{-3a^2b^2}{4cd} = \frac{ab^2}{2c^2} \cdot \frac{4cd}{-3a^2b^2}$$

$$= -\frac{ab^2 \cdot 4cd}{2c^2 \cdot 3a^2b^2} = -\frac{2d}{3ac}$$

例 3.79(Example 3.79)

计算(Calculate)

$$\frac{2x-6}{4-4x+x^2} \div (x+3) \cdot \frac{x^2+x-6}{3-x}$$

解(Solution):

$$\frac{2x-6}{4-4x+x^2} \div (x+3) \cdot \frac{x^2+x-6}{3-x}$$

$$= \frac{2(x-3)}{(x-2)^2} \cdot \frac{1}{x+3} \cdot \frac{(x+3)(x-2)}{-(x-3)}$$

$$= -\frac{2(x-3)(x+3)(x-2)}{(x-2)^2(x+3)(x-3)} = -\frac{2}{x-2}$$

2. 分式的乘方(Power of Fractions)

分式的乘方是把分子、分母各自乘方,即

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (n \text{ 为正整数})$$

The power of a fraction is to multiply the exponents of its numerator and denominator, respectively, that is

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (n \text{ is any positive integer})$$

例 3.80(Example 3.80)

计算(Calculate)

(1) $\left(\frac{2a^2b}{-c^3}\right)^3$

(2) $\left(-\frac{x^2}{y}\right)^2 \cdot \left(-\frac{y^2}{x}\right)^3 \div \left(-\frac{y}{x}\right)^4$

解(Solution):

(1) $\left(\frac{2a^2b}{-c^3}\right)^3 = \frac{2^3(a^2)^3 \cdot b^3}{(-c^3)^3} = \frac{8a^6b^3}{-c^9}$

(2) $\left(-\frac{x^2}{y}\right)^2 \cdot \left(-\frac{y^2}{x}\right)^3 \div \left(-\frac{y}{x}\right)^4$

$= \frac{x^4}{y^2} \cdot \left(-\frac{y^6}{x^3}\right) \div \frac{y^4}{x^4} = \frac{x^4}{y^2} \cdot \left(-\frac{y^6}{x^3}\right) \cdot \frac{x^4}{y^4}$

$= -\frac{x^4 \cdot y^6 x^4}{y^2 x^3 y^4} = -\frac{x^8 y^6}{y^6 x^3} = -x^5$

3. 分式的加减(Addition and Subtraction of Fractions)

1) 最简公分母(Least Common Denominator)

分式中各分母所有因式的最高次幂的积叫做最简公分母.

The product of all the different prime factors in the denominators of the fractions, each taken the greatest number of times it occurs in any of the denominators, is called the least common denominator.

2) 通分(Reduction of Fractions to a Common Denominator)

把 n 个异分母的分式分别化成与原来的分式相等的同分母的分式,这个过程叫做通分.通分时,每个分式的分子和分母均乘

以一个相应的值.

The process of reducing fractions that have no common denominator to those that have a common denominator is called the reduction of fractions to a common denominator. To do this, multiply the numerator and denominator of each fraction by an appropriate value.

例 3.81 通分 (1) $\frac{y}{(1)\ 2x}$ (2) $\frac{x}{3y^2}$ (3) $\frac{1}{4xy}$

解:

∵ 最简公分母是 $12xy^2$

∴ (1) $\frac{y}{2x} = \frac{y \cdot 6y^2}{2x \cdot 6y^3} = \frac{6y^2}{12xy^2}$

(2) $\frac{x}{3y^2} = \frac{x \cdot 4x}{3y^2 \cdot 4x} = \frac{4x^2}{12xy^2}$

(3) $\frac{1}{4xy} = \frac{1 \cdot 3y}{4xy \cdot 3y} = \frac{3y}{12xy^2}$

Example 3.81 Reduce the following fractions to a common denominator.

(1) $\frac{y}{2x}$ (2) $\frac{x}{3y^2}$ (3) $\frac{1}{4xy}$

Solution

Since the least common denominator is $12xy^2$,

(1) $\frac{y}{2x} = \frac{y \cdot 6y^2}{2x \cdot 6y^2} = \frac{6y^3}{12xy^2}$

(2) $\frac{x}{3y^2} = \frac{x \cdot 4x}{3y^2 \cdot 4x} = \frac{4x^2}{12xy^2}$

(3) $\frac{1}{4xy} = \frac{1 \cdot 3y}{4xy \cdot 3y} = \frac{3y}{12xy^2}$

3) 同分母的分式加减法(Addition and Subtraction of Fractions with Common Denominators)

同分母的分式相加减,分母不变,把分子相加减,即

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

In addition or subtraction of the fractions with common denominators, add or subtract their numerators without changing their denominators, that is

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

例 3.82(Example 3.82)

计算(Calculate)

$$\frac{x+3y}{x^2-y^2} - \frac{x+2y}{x^2-y^2} + \frac{2x-3y}{x^2-y^2}$$

解(Solution):

$$\frac{x+3y}{x^2-y^2} - \frac{x+2y}{x^2-y^2} + \frac{2x-3y}{x^2-y^2}$$

$$= \frac{(x+3y)-(x+2y)+(2x-3y)}{x^2-y^2} = \frac{2x-2y}{x^2-y^2}$$

$$= \frac{2(x-y)}{(x+y)(x-y)} = \frac{2}{x+y}$$

例 3.83(Example 3.83)

计算(Calculate)

$$-\frac{m-2n}{n-m} + \frac{n}{m-n} - \frac{2m}{n-m}$$

解(Solution):

$$-\frac{m-2n}{n-m} + \frac{n}{m-n} - \frac{2m}{n-m}$$

$$= \frac{m-2n}{m-n} + \frac{n}{m-n} + \frac{2m}{m-n}$$

$$= \frac{m-2n+n+2m}{m-n} = \frac{3m-n}{m-n} = 1$$

4) 异分母的分式加减法(Addition and Subtraction of Fractions

with Different Denominators)

异分母的分式相加减，先通分，变为同分母的分式，然后再加减，即

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$

In addition and subtraction of fractions with different denominators, rewrite the fractions so that they do have a common denominator and then do the addition and subtraction, that is

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$

例 3.84(Example 3.84)

计算(Calculate)

$$\frac{5}{6ab} - \frac{2}{3ac} + \frac{3}{4abc}$$

解(Solution)：

$$\frac{5}{6ab} - \frac{2}{3ac} + \frac{3}{4abc}$$

$$= \frac{10c}{12abc} - \frac{8b}{12abc} + \frac{9}{12abc}$$

$$= \frac{10c - 8b + 9}{12abc}$$

例 3.85(Example 3.85)

计算(Calculate) $a + 2 - \frac{4}{2 - a}$

解(Solution)：

$$a + 2 - \frac{4}{2 - a} = \frac{a + 2}{1} + \frac{4}{a - 2}$$

$$= \frac{a^2 - 4 + 4}{a - 2} = \frac{a^2}{a - 2}$$

4. 分式的混合运算(Mixed Operation of Fractions)

分式的加、减、乘、除混合运算仍是先进行乘除运算,再进行加减运算,遇有括号,先算括号内的.

The mixed operation of fractions in addition, subtraction, multiplication and division is still to do multiplication and division first and then addition and subtraction. If there are brackets, first do the operation within the brackets.

例 3.86(Example 3.86)

计算(Calculate) $\dfrac{1}{x+1}-\dfrac{x+3}{x^2-1}\cdot\dfrac{x^2-2x+1}{x^2+4x+3}$

解(Solution):

$$\frac{1}{x+1}-\frac{x+3}{x^2-1}\cdot\frac{x^2-2x+1}{x^2+4x+3}$$

$$=\frac{1}{x+1}-\frac{(x+3)}{(x+1)(x-1)}\cdot\frac{(x-1)^2}{(x+1)(x+3)}$$

$$=\frac{1}{x+1}-\frac{x-1}{(x+1)^2}=\frac{x+1-(x-1)}{(x+1)^2}=\frac{2}{(x+1)^2}$$

例 3.87(Example 3.87)

计算(Calculate) $\left(\dfrac{x+2}{x^2-2x}-\dfrac{x-1}{x^2-4x+4}\right)\div\dfrac{x-4}{x}$

解(Solution):

$$\left(\frac{x+2}{x^2-2x}-\frac{x-1}{x^2-4x+4}\right)\div\frac{x-4}{x}$$

$$=\left[\frac{x+2}{x(x-2)}-\frac{x-1}{(x-2)^2}\right]\cdot\frac{x}{x-4}$$

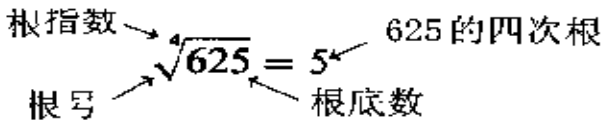
$$=\frac{(x+2)(x-2)-x(x-1)}{x(x-2)^2}\cdot\frac{x}{x-4}$$

$$=\frac{x^2-4-x^2+x}{(x-2)^2(x-4)}=\frac{1}{(x-2)^2}$$

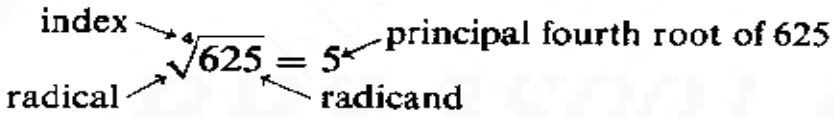
二、根式(Radicals)

(一) 定义(Definitions)

含有根号的代数式称为根式，记为$\sqrt[n]{b}$. $\sqrt[n]{\ }$ 被称为根号，n 被称为根指数或根(n 为大于 1 的正整数)，b 称为根底数或被开方式. 当 $b \geqslant 0$ 时，$\sqrt[n]{b}$ 叫做算术根.



Algebraic expressions containing radicals are called radical expressions which are written as $\sqrt[n]{b}$. $\sqrt[n]{\ }$ is the radical sign(or simply the radical), n , which is any positive integer greater than one, is called the index or root and b under the radical is called the base or radicand. When $b \geqslant 0$, $\sqrt[n]{b}$ is called an arithmetical root.



例 3.88(Example 3.88)

以下各式均为根式.

(Each of the following expressions is a radical one)

$\sqrt[3]{1080}$; $\sqrt[5]{128x^{-6}y^{12}}$; $\dfrac{4x}{\sqrt[6]{32x^2}}$

根式还可记为：$\sqrt[n]{b} = b^{1/n}$

当 n 是偶自然数时，则 b 是非负数；当 n 是奇自然数时，则 b 为任意实数，即

$$\sqrt[n]{-b} = -\sqrt[n]{b}.$$

A radical is defined by $\sqrt[n]{b} = b^{1/n}$

If n is an even natural number, then b is a nonnegative real number; if n is an odd natural number, then b is any real number, namely

$$\sqrt[n]{-b} = -\sqrt[n]{b}.$$

例 3.89(Example 3.89)

简化下列根式(Simplify the following radicals):

$$\frac{\sqrt[6]{4x^{14}}}{\sqrt[6]{32x^2}} = \frac{2^{2/6}x^{14/6}}{2^{5/6}x^{2/6}} = 2^{-3/6}x^{12/6} = \frac{x^2}{2^{1/2}} = \frac{x^2}{\sqrt{2}}.$$

(二) 基本性质(Basic Properties)

一个数或一个式的 n 次根式的 n 次方或 n 次方开 n 次根式都等于这个数或式,记作:

$$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a \quad (a \geqslant 0)$$

Either the nth power of a radical expression to the nth power or the nth root of a real number to its nth power equals the real number or the expression, denoted by

$$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a \quad (a \geqslant 0)$$

当 a 在实数范围内,$\sqrt{a^2} = a$ 不是恒等式.

If a is within real numbers, $\sqrt{a^2} = a$ is not identical equation.

例 3.90(Example 3.90)

$$\sqrt{a^2} = |a| \text{ or } \sqrt{a^2} = \begin{cases} a & (a \geqslant 0) \\ -a & (a < 0) \end{cases}$$

例 3.91(Example 3.91)

$$\frac{\sqrt{x^2}}{x} = \frac{|x|}{x} = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \\ \text{无根(no root)} & (x = 0) \end{cases}$$

(三) 变形规则(Laws of Radicals)

1. 基本规则(Basic Law)

约去被开方数或式与根指数之间的公约数,根式的值不变,即:

$$\sqrt[np]{a^{mp}} = \sqrt[n]{a^m} \qquad (a \geqslant 0)$$

If we eliminate the common factor between the power of the radicand or expression and the index of the radical and the value of the radical expression remains unchangeable, namely

$$\sqrt[np]{a^{mp}} = \sqrt[n]{a^m} \qquad (a \geqslant 0)$$

利用这个规则,可以把根式约简,或根据需要化根指数不同的根式(异次根式)为根指数相同的根式(同次根式).

Using this law, we can simplify a radical expression or change radical expressions with different indexes of radicals (different radicals) to radical expressions with a like index of the radical (like radicals).

例 3.92(Example 3.92)

简化下列根式(Simplify the following radical expressions).

(1) $\sqrt[3]{a^6}$ (2) $\sqrt[6]{(-3)^2x^4}$

解(Solution): (1) $\sqrt[3]{a^6} = a^2$

(2) $\sqrt[6]{(-3)^2x^4} = \sqrt[6]{3^2x^4} = \sqrt[3]{3x^2}$

例 3.93(Example 3.93)

化 $\sqrt[3]{-5}$ 和 $\sqrt{3}$ 为同类根式

(Change $\sqrt[3]{-5}$ and $\sqrt{3}$ into like radicals).

解(Solution):

$$\sqrt[3]{-5} = -\sqrt[3]{5} = -\sqrt[6]{5^2} = \sqrt[6]{25}$$

$$\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$$

2. 乘法规则(Law of Multiplication)

积的 n 次根,等于积中各因式的 n 次根的积,即

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad (a \geqslant 0,\ b \geqslant 0)$$

The nth root of a product equals the product of the nth root of every factor, that is

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad (a \geqslant 0,\ b \geqslant 0)$$

例 3.94(Example 3.94)

简化 (Simplify)

(1) $\sqrt[3]{1080}$ (2) $\sqrt{32}$ (3) $\dfrac{4x}{\sqrt[6]{32x^2}}$

解(Solution):

(1) $\sqrt[3]{1080} = \sqrt[3]{8 \times 27 \times 5} = \sqrt[3]{2^3 \times 3^3 \times 5}$

$= 2 \cdot 3 \cdot \sqrt[3]{5} = 6\sqrt[3]{5}$

(2) $\sqrt{32} = \sqrt{2^5} = \sqrt{2^4 \cdot 2} = \sqrt{(2^2)^2 \cdot 2} = 2^2\sqrt{2} = 4\sqrt{2}$

(3) $\dfrac{4x}{\sqrt[3]{32x^2}} = \dfrac{4x}{\sqrt[6]{2^5x^2}} \quad (x \neq 0)$

$$= \frac{4x\sqrt[6]{2x^4}}{\sqrt[6]{2^5x^2} \cdot \sqrt[6]{2x^4}} = \frac{4x\sqrt[6]{2x^4}}{\sqrt[6]{2^6x^6}} = \frac{4x\sqrt[6]{2x^4}}{2 \mid x \mid}$$

$$= \begin{cases} 2\sqrt[3]{2x^4} & x \text{ 为正(If } x \text{ is positive)} \\ -2\sqrt[6]{2x^4} & x \text{ 为负(If } x \text{ is negative)} \end{cases}$$

（四）除法规则（Law of Division）

商的 n 次根等于被除式的 n 次根除以除式的 n 次根，即

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (a \geqslant 0,\ b \geqslant 0)$$

The nth root of a quotient equals the nth root of a dividend divided by the nth root of a divisor, that is

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (a \geqslant 0,\ b \geqslant 0)$$

把分母中的根式化去，叫做分母有理化.

The process to eliminate the radical in a denominator is called rationalizing the denominator.

例 3.95（Example 3.95）

将下列根式分母有理化.

（Rationalize the denominators of the following radicals）

$$(1)\sqrt{\frac{2b}{3a}} \quad (2)\sqrt[3]{\frac{3}{4}} \quad (3)\sqrt{\frac{a-b}{(a+b)}}$$

解（Solution）：

$$(1)\sqrt{\frac{2b}{3a}} = \sqrt{\frac{2b \cdot 3a}{3a \cdot 3a}} = \frac{\sqrt{6ab}}{3a}$$

$$(2)\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{3 \times 2}{4 \times 2}} = \frac{\sqrt[3]{6}}{\sqrt[3]{8}} = \frac{1}{2}\sqrt[3]{6}$$

$$(3)\sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{(a-b)(a+b)}{(a+b)^2}} = \frac{\sqrt{a^2-b^2}}{a+b}$$

（五）乘方规则（Law of Power）

n 次根式的 m 次方，等于被开方数或式的 m 次方开 n 次根，可

记为底数的$\frac{m}{n}$次方，即

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

The mth power of the nth radical equals the nth root of the radical or radical expression to the mth power, which can be written as the $\frac{m}{n}$ th power of the base, that is

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

例 3.96(Example 3.96)

(1) $\sqrt[3]{1080}$　　(2) $\sqrt{32}$

解(Solution)：

(1) $\sqrt[3]{1080} = \sqrt[3]{2^3 \cdot 3^3 \cdot 5} = 2 \cdot 3 \cdot \sqrt[3]{5} = 6\sqrt[3]{5}$

(2) $\sqrt{32} = 2^{\frac{5}{2}} = 2^{2\frac{1}{2}} = 2^{2+\frac{1}{2}} = 2^2 \cdot 2^{\frac{1}{2}} = 4\sqrt{2}$

(六) 根式的运算和化简

(Operation and Simplification of Radicals)

1. 最简根式(Simplest Radicals)

被开方数的每一个因式的指数都小于根指数且分母不含根式的根式叫做最简根式.

The radical expression in which the index of every factor of the radicand is less than the index of the radical and which is free of denominator is called the simplest radical or radical expression.

例 3.97(Example 3.97)

将下列根式变成最简根式：

Change the following radicals into the simplest radicals:

(1) $\sqrt[4]{a^2b^2} = \sqrt{ab}$

(2) $\sqrt{a^3b} = a\sqrt{ab}$

(3) $\sqrt{\frac{b}{a}} = \frac{1}{a}\sqrt{ab}$

$\sqrt{ab}$, $a\sqrt{ab}$ 和$\frac{1}{a}\sqrt{ab}$ 都为最简根式，$\sqrt[4]{a^2b^2}$, $\sqrt{a^3b}$ 和$\sqrt{\frac{b}{a}}$ 都不是最简根式.

$\sqrt{ab}$, $a\sqrt{ab}$ and $\frac{1}{a}\sqrt{ab}$ are all the simplest radicals, while $\sqrt[4]{a^2b^2}$, $\sqrt{a^3b}$ and $\sqrt{\frac{b}{a}}$ are not the simplest ones.

2. 同类根式(Like Radicals)

根指数和被开方数都相同的根式叫做同类根式， 如$\sqrt{ab}$, $a\sqrt{ab}$ 和$\frac{1}{a}\sqrt{ab}$ 就是同类根式.

The radical expressions containing the same index and radicand are called like or similar radical expressions. For example, $\sqrt{ab}$, $a\sqrt{ab}$ and $\frac{1}{a}\sqrt{ab}$ are like or similar expressions.

3. 根式的运算和化简法则

(Rules of Operation and Simplification of Radicals)

加减法的基本规则是去括号和合并同类根式;乘法是运用分配律及乘法公式;除法相当于分式. 根式相乘除时,必须先把根式化成同类根式，化简的结果是各项都必须是最简根式.

The basic rule for addition and subtraction of radicals is to eliminate their signs of grouping and combine the similar radicals; for multiplication of radicals, apply the distributive properties and the formulas of special products; for division, just like fractions, simplify radical expressions into similar expressions, and as a result, each term must be the simplest radical expression.

例 3.98(Example 3.98)

$$\sqrt[5]{128x^{-6}y^{12}} = \sqrt{\frac{2^7y^{12}}{x^6}}$$

$$= \sqrt{\frac{(2)^5(2)^2(y^2)^5 \cdot y^2}{x^5 \cdot x} \cdot \frac{x^4}{x^4}}$$

$$= \sqrt{(\frac{2y^2}{x^2})^5 \cdot \frac{(2)^2y^2x^4}{1}} = \frac{2y^2}{x^2}\sqrt[5]{4x^4y^2}$$

例 3.99(Example 3.99)

$$\frac{\sqrt{x}}{x} + \sqrt{\frac{2}{3}} + \frac{2\sqrt[3]{18}}{3} + \frac{1}{\sqrt{x}}$$

$$= \frac{\sqrt{x}}{x} + \sqrt{\frac{2}{3} \cdot \frac{3^2}{3^2}} + \frac{2\sqrt[3]{18}}{3} + \frac{1}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{\sqrt{x}}{x} + \frac{\sqrt[3]{18}}{3} + \frac{2\sqrt[3]{18}}{3} + \frac{\sqrt{x}}{x} = \frac{2\sqrt{x}}{x} + \sqrt[3]{18}$$

(七) 分数指数幂(Powers of Fractional Exponents)

数或式的正分数$\frac{m}{n}$次幂,是指这个数的 m 次幂的n 次根. 数或式的负分数 $-\frac{m}{n}$ 次幂, 是指这个数的 m 次幂的n 次根的倒数, 即

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (a \geqslant 0,\ m, n \text{ 是正整数}),$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}} \quad (a > 0,\ m, n \text{ 是正整数}).$$

The $\frac{m}{n}$ th power of a real number or an expression means the nth root of the number to the mth power if the fraction $\frac{m}{n}$ is a positive rational number. The minus $\frac{m}{n}$ th power of a real number is the reciprocal of the

nth root of the number to the mth power if the fraction $\frac{m}{n}$ is a negative rational number, that is

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (a \geqslant 0,\ m, n \text{ are positive integers}),$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}} \quad (a > 0,\ m, n \text{ are positive integers}).$$

对于分数指数幂，它的运算仍符合幂运算的指数律.

$a^x \cdot a^y = a^{x+y}$　　$(a^x)^y = a^{xy}$　　$(ab)^x = a^x \cdot b^x$

其中 x, y 可以是正负整数和零，也可以是正负分数.

For the power of a fractional exponent. its operation also follows the laws of exponents for power operation.

$a^x \cdot a^y = a^{x+y}$　　$(a^x)^y = a^{xy}$　　$(ab)^x = a^x \cdot b^x$

in which x, y can be either positive, negative integers or zero, as well as positive and negative fractions.

例 3.100(Example 3.100)

简化下例根式(Simplify the following radical expressions).

(1) $\sqrt{125} \cdot \sqrt[4]{5}$

$$= \sqrt{5^3}\sqrt[4]{5} = 5^{\frac{3}{2}} \cdot 5^{\frac{1}{4}} = 5^{\frac{3}{2}+\frac{1}{4}} = 5^{\frac{7}{4}} = 5^{1+\frac{3}{4}}$$

$$= 5\sqrt[4]{5^3}$$

(2) $\dfrac{a\sqrt[5]{a^3}}{\sqrt{a} \cdot \sqrt[10]{a^7}} = a \cdot a^{\frac{3}{5}} \cdot a^{-\frac{1}{2}} \cdot a^{-\frac{7}{10}} = a^{\left(1+\frac{3}{5}-\frac{1}{2}-\frac{7}{10}\right)}$

$$= a^{\left(\frac{2}{5}\right)} = \sqrt[5]{a^2}$$

(3) $\sqrt[4]{\left(\dfrac{16m^{-4}}{81n^4}\right)^3} = \left(\dfrac{2^4 m^{-4}}{3^4 n^4}\right)^{\frac{3}{4}} = \dfrac{2^3 m^{-3}}{3^3 n^3} = \dfrac{8}{27m^3n^3}$

例 3.101(Example 3.101)

化下列双根式为单根式:

Change the following double radicals into single radicals:

(1) $\sqrt{\sqrt[3]{3a^3}} = \sqrt[6]{3a^3}$

(2) $\sqrt{2a\sqrt[3]{x}} = \sqrt{\sqrt[3]{8a^3x}} = \sqrt[6]{8a^3x}$

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第四章　方程和不等式的表达 (Equations and Inequalities)

§4.1　方程的表达(Equations)

一、几个基本定义(Several Basic Definitions)

(一) 方程(Equations)

含有未知数的等式称为方程.

Any statement that two expressions containing variables are equal is called an equation.

例 4.1(Example 4.1)　$5-2x=1$;　$x-2y=6$.

(二) 方程的解(Solution of Equations)

使方程左右两边值相等的未知数的解,叫做方程的解,只有一个未知数的方程的解,叫做根.　例 方程 $5x-7=8$ 的解或根为 $x=3$.

The solution of an unknown that makes both sides of an equation identical is called the solution of the equation. The solution of an equation which has only one unknown is called the root of the equation. For instance, the solution or root of the equation $5x-7=8$ is $x=3$.

(三) 单变量的代数方程(Algebraic Equations with One Variable)

一个方程如果有下列形式

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

其中 $a_n \neq 0$, $a_{n-1}, a_{n-2}, \ldots a_0$是有理数，$n \in N$，则称此方程为变量 x 的代数方程.

If an equation has the following forms

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

where $a_n \neq 0$, $a_{n-1}, a_{n-2}, \ldots a_0$ are rational numbers, $n \in N$, then we call this type of equation the algebraic equation containing variable x.

1. 线性方程(Linear Equations)

方程中变量的最高次数仅为一次的方程称为线性方程.

An equation whose highest degree of variable(s) is only one is called the linear equation.

例 4.2(Example 4.2) $ax + b = 0$, $a \neq 0$, $x \in R$

2. 二次方程(Quadratic Equations)

方程中变量的最高次数为 2 次的方程称为二次方程，也称为一元二次方程.

An equation whose highest degree of variable(s) is two is called the quadratic equation or the second-degree equation.

例 4.3(Example 4.3) $ax^2 + bx + c = 0$, $a \neq 0$

3. n 次方程(nth-degree Equations)

方程中变量的最高次数为 n 次的方程称为 n 次方程.

The equations containing degree n are called the nth-degree equations.

例 4.4(Example 4.4) $Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots = 0$, $A \neq 0$

n 次代数方程有 n 个实根或复根. 例

$$(y+1)^3+3(y+1)^2-4(y+1)-4=0$$

为一元三次方程,应有 3 个根. 这 3 个根或为实根或为复根.

An nth-degree algebraic equation has n real roots or complex roots. For example,

$$(y+1)^3+3(y+1)^2-4(y+1)-4=0$$

is a third-degree equation and should have three roots which are either real roots or complex roots.

(四) 方程组(Systems of Equations)

由多个变量、多个方程组成的一组方程称为方程组,当方程组中各变量的最高次数为 1 时,称为线性方程组.

A collection of two or more equations with two or more variables is called a system of equations. If the highest degree of each variable is 1, the system is called a system of linear equations.

1. 二元线性方程组

(Systems of Linear Equations with Two Variables)

由两个变量组成,且各变量的最高次数为一次的 2 个方程组成的方程组称为二元线性方程组.

A collection of two equations with two variables whose highest degree is 1 is called a system of linear equations with two variables.

例 4.5(Example 4.5)

$$\begin{cases} a_{11}x+a_{12}y=b_1 \\ a_{21}x+a_{22}y=b_2 \end{cases}$$

2. 三元线性方程组

(Systems of Linear Equations with Three Variables)

由 3 个变量,且最高次数为 1 的三个方程组成的方程组称为三元线性方程组.

A collection of three equations with three variables whose highest degree is 1 is called the system of linear equations with three variables.

例 4.6(Example 4.6)

$$\begin{cases} 3x + 3y + z = 7 \\ 3x + y + 3z = 15 \\ x + 3y + 3z = 13 \end{cases}$$

3. n 个变量的线性方程组

(Systems of Linear Equations with n Variables)

有 n 个变量,且最高次数为 1 的 n 个线性方程组成的方程组为 n 个变量的线性方程组.

A collection of n equations with n variables whose highest degree is 1 is called the system of n linear equations with n variables.

例 4.7(Example 4.7)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = c_3 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n \end{cases}$$

二、单变量方程的求解
(Solution for Equations with One Variable)

(一) 一次方程的求解(Solution for the First-Degree Equations)

求解的步骤为:

① 去分母(用最简公分母乘方程两边所有项);

② 去括号(括号前是负号时,括号内各项都变号);

③ 移项(要变号);

④ 合并同类项;

⑤ 方程两边同除以未知数系数;

⑥ 代入原方程,进行检验.

The procedure for solving a first-degree equation is as follows:

① Eliminate denominators (multiply each term on both sides of the equation by the least common denominator);

② Eliminate signs of grouping (if the sign before the sign of grouping is negative, change the sign of every term in the sign of grouping);

③ Remove terms (change signs);

④ Combine similar terms;

⑤ Divide both sides of the equation by the coefficient of an unknown;

⑥ Check your answer by bringing it to the original equation.

例 4.8 $29 - 2x = 15x - 5$

解: $29 - 2x = 15x - 5$

$29 - 2x - 15x = -5$ 两边分别减去 $15x$.

$29 - 17x = -5$ 合并同类项.

$-17x = -34$ 两边分别减去 29.

$$17x = 34 \qquad \text{两边分别乘以 } -1.$$

$$x = \frac{34}{17} \qquad \text{两边分别除以 17.}$$

$$x = 2$$

Example 4.8 $29 - 2x = 15x - 5$

Solution $29 - 2x = 15x - 5$

$29 - 2x - 15x = -5$ Subtract $15x$ from both sides.

$29 - 17x = -5$ Combine like terms.

$-17x = -34$ Subtract 29 from both sides.

$17x = 34$ Multiply both sides hy -1.

$x = \frac{34}{17}$ Divide both sides by 17.

$x = 2$

例 4.9 求方程 $ax + 4c = b - 2x$ 的 x 解.

解: $ax + 4c = b - 2x$

$ax + 2x + 4c = b$ 两边加 $2x$,使含 x 的项均在左边.

$ax + 2x = b - 4c$ 两边分别减去 $4c$,使不含 x 的项均在右边.

$(a + 2)x = b - 4c$ 用乘法分配定律.

设 $a + 2 \neq 0$,方程两边均除以 $a + 2$,求得以下解

$$x = \frac{b - 4c}{a + 2}, \qquad a \neq -2$$

Example 4.9 Solve the equation $ax + 4c = b - 2x$ for x.

Solution $ax + 4c = b - 2x$

$ax + 2x + 4c = b$ Add $2x$ to both sides so that all terms containing x are on the left side.

$ax + 2x = b - 4c$ Subtract $4c$ from both sides so that all terms not containing x are on the right side

$(a + 2)x = b - 4c$ Use the distributive property.

Now, provided that $a + 2 \neq 0$, we can divide both sides of the last equation by $a + 2$, to obtain the solution

$$x = \frac{b - 4c}{a + 2} \qquad \text{for } a \neq -2$$

(二) 二次方程的求解(Solution for Quadratic Equations)

一元二次方程的一般形式为

$$ax^2 + bx + c = 0, \quad a \neq 0$$

A second-degree or quadratic equation is written in standard form as

$$ax^2 + bx + c = 0, \quad a \neq 0$$

1. 因式分解法(Factoring)

如果两个因式乘积为0,则两个因式中至少有一个为0,即:如果 $AB = 0$,则 $A = 0$ 或 $B = 0$,这个性质可以用来求一元二次方程.

If the product of two factors is zero, then at least one of the factors is zero. Namely, $AB = 0$, if and only if $A = 0$ or $B = 0$. This property can be used to solve a quadratic equation.

因式分解法的求解步骤:

① 将方程非零项全部写在方程左边;

② 用因式相乘的方式表达方程;

③ 设每一个因式等于0;

④ 解每个线性方程;

⑤ 写出二次方程的一组解集.

Solution of quadratic equations by factoring:

① Rewrite all nonzero terms on the left side of the equation;

② Factor the expression;

③ Let each of the factors equal zero;

④ Solve each of the linear equations;

⑤ Write the solution set which is the union of the solution sets of the linear equations.

例 4.10(Example 4.10)　$x^2 = 2x + 15$

$x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$

$x + 3 = 0$　或(or) $x - 5 = 0$

$\therefore \quad x = -3$　　$x = 5$

2. 配方法(Completing the Square)

配方法基于平方根的性质,如果 $P^2 = Q$,则 $P = \pm\sqrt{Q}$.

Completing the square is a method which depends on the square root property. If $P^2 = Q$, then $P = \pm\sqrt{Q}$.

例 4.11　用配方法和因式分解法求解方程 $x^2 = 4$.

解:

配方法	因式分解法
$x^2 = 4$	$x^2 - 4 = 0$
$x = \pm\sqrt{4}$	$(x + 2)(x - 2) = 0$
$x = \pm 2$	$x = -2$　或　$x = 2$

Example 4.11　Compare the method of square root property and that of factoring in solving the equation $x^2 = 4$.

Solution

Square Root Property	Factoring
$x^2 = 4$	$x^2 - 4 = 0$
$x = \pm\sqrt{4}$	$(x + 2)(x - 2) = 0$
$x = \pm 2$	$x = -2$　or　$x = 2$

配方法求解步骤：

① 方程两边各加以常数项；

② 方程两边同除以二次项系数，使二次项系数为 1；

③ 将常数项移到方程右边；

④ 方程两边各加一次项系数一半的平方，然后配方；

⑤ 开方求解.

Procedures for completing the square:

① Add the constant term to both sides of the equation;

② Divide both sides of the equation by the coefficient of the second-degree term so that the coefficient is 1;

③ Move the constant to the right side of the equation;

④ Add the square of half the coefficient of the first-degree term to both sides of the equation, and then complete the square;

⑤ Extract the square root for the solution.

例 4.12　解方程　$x^2 = 2x + 15$

解：

$$x^2 = 2x + 15$$

$$x^2 - 2x = 15 \qquad \text{将常数项放在等号右边.}$$

$$x^2 - 2x + 1 = 15 + 1 \qquad \text{方程两边各加一次系数一半的平方.}$$

$$(x - 1)^2 = 16 \qquad \text{配方.}$$

$$x - 1 = \pm 4 \qquad \text{开方.}$$

$$x = 1 \pm 4 \qquad \text{求解 } x.$$

$$x_1 = 1 + 4 \quad \text{或} \quad x_2 = 1 - 4$$

$$x_1 = 5 \quad \text{或} \quad x_2 = -3$$

$$\{5, -3\}$$

Example 4.12　Solve for the equation $x^2 = 2x + 15$

Solution

$$x^2 = 2x + 15$$

$x^2 - 2x = 15$ Place the constant term on the right side.

$x^2 - 2x + 1 = 15 + 1$ Complete the square by adding the square of half the coefficient of x to both sides.

$(x - 1)^2 = 16$ Complete the square.

$x - 1 = \pm 4$ Use the square root property.

$x = 1 \pm 4$ Solve for x .

$x_1 = 1 + 4$ or $x_2 = 1 - 4$

$x_1 = 5$ or $x_2 = -3$

$\{5, -3\}$

例 4.13 解方程 $4x^2 - 4x - 7 = 0$

解： $4x^2 - 4x - 7 = 0$

$4x^2 - 4x = 7$ 将常数项放在等式右边.

$x^2 - x = \frac{7}{4}$ 方程两边同除以 4 使 x^2 的系数为 1.

$x^2 - x + \left(\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right)^2 + \frac{7}{4}$ 两边各加 x 系数一半的平方.

$\left(x - \frac{1}{2}\right)^2 = 2$ 配方.

$x - \frac{1}{2} = \pm\sqrt{2}$ 开方.

$x = \frac{1 \pm 2\sqrt{2}}{2}$ 解 x.

$\left\{\frac{1 + 2\sqrt{2}}{2}, \frac{1 - 2\sqrt{2}}{2}\right\}$

Example 4.13 Solve for the equation $4x^2 - 4x - 7 = 0$

Solution $4x^2 - 4x - 7 = 0$

$4x^2 - 4x = 7$ Move the constant term to the right side.

$x^2 - x = \frac{7}{4}$ Divide by 4 to obtain 1 as the coefficient of x^2.

$x^2 - x + \left(\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right)^2 + \frac{7}{4}$ Add the square of half the coefficient of x to both sides.

$\left(x - \frac{1}{2}\right)^2 = 2$ Complete the square.

$x - \frac{1}{2} = \pm\sqrt{2}$ Extract the square root.

$x = \frac{1 \pm 2\sqrt{2}}{2}$ Solve for x.

$\left\{\frac{1+2\sqrt{2}}{2}, \frac{1-2\sqrt{2}}{2}\right\}$

3. 公式求解法(Quadratic Formulas)

配方法常常是很麻烦的,然而对于任意一个二次方程 $ax^2 + bx + c = 0$, $a \neq 0$,可以用配方法以 a, b, c 为系数推出一个对于 x 的二次方程公式.该公式能解所有的一元二次方程,包括那些无法用因式分解法求解的方程.

The process of completing the square is often cumbersome. However, if any quadratic $ax^2 + bx + c = 0$, $a \neq 0$, is concerned, completing the square can be used to derive a formula for x in terms of the coefficients a, b and c. The formula can then be used to solve all quadratics, even those that are nonfactorable.

例 4.14 $ax^2 + bx + c = 0$

$$ax^2 + bx = -c \qquad \text{移常数项.}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{两边除以 } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2}\right) - \frac{c}{a} \qquad \text{方程两边同加 } x \text{ 系数一半的平方.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{配方.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{开方.}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{求出 } x.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

因此,如果 $ax^2 + bx + c = 0$, $a \neq 0$, 则:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 4.14 $ax^2 + bx + c = 0$

$$ax^2 + bx = -c \qquad \text{Move the constant term.}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divided by } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2}\right) - \frac{c}{a} \qquad \text{Add the square of half the coefficient of } x \text{ to both sides.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Complete the square.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Extract the square root.}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, if $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

例 4.15(Example 4.15)

求解(Solve) $4x^2 - 4x - 7 = 0$

解(Solution): $a = 4 \quad b = -4 \quad c = -7$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-7)}}{2(4)}$$

$$x = \frac{4 \pm \sqrt{128}}{8}$$

$$x = \frac{4 \pm 8\sqrt{2}}{8}$$

$$x = \frac{4(1 \pm 2\sqrt{2}}{4 \cdot 2}$$

$$x = \frac{1 \pm 2\sqrt{2})}{2}$$

$\therefore \quad x_1 = \frac{1 + 2\sqrt{2}}{2}; \quad x_2 = \frac{1 - 2\sqrt{2}}{2}$

例 4.16(Example 4.16)

求解(Solve) $2x^2 - 2x + 5 = 0$

解(Solution): $a = 2 \quad b = -2 \quad c = 5$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(5)}}{2(2)}$$

$$x = \frac{2 \pm \sqrt{-36}}{4}$$

$$x = \frac{2 \pm 6\mathrm{i}}{4}$$

$$x = \frac{1 \pm 3\mathrm{i}}{2}$$

$$\therefore \quad x_1 = \frac{1 + 3\mathrm{i}}{2}, \quad x_2 = \frac{1 - 3\mathrm{i}}{2}$$

4. 根的判别式(Discriminant of the Quadratic Form)

如果 $ax^2 + bx + c = 0$, $a \neq 0$, 则 $b^2 - 4ac$ 称为方程 $ax^2 + bx + c = 0$ 的根的判别式.

当 $b^2 - 4ac < 0$ 时, 方程没有实数根.

当 $b^2 - 4ac = 0$ 时, 方程有两个相等的实数根.

当 $b^2 - 4ac > 0$ 时, 方程有两个不相等的实数根.

If $ax^2 + bx + c = 0$, $a \neq 0$, then $b^2 - 4ac$ is called the discriminant of the quadratic form.

If $b^2 - 4ac < 0$, there are no real roots.

If $b^2 - 4ac = 0$, there are two equal real roots.

If $b^2 - 4ac > 0$, there are two unequal real roots.

找出例 4.17 ~ 4.19 中每个二次方程的判别式,标出实数根的数量.

Find the discriminant of each quadratic in examples 4.17 ~ 4.19, and specify the number of real roots.

例 4.17 $x^2 = 2x + 15$

解: $x^2 - 2x - 15 = 0$ 变为标准式.

$a = 1 \quad b = -2 \quad c = -15$

$b^2 - 4ac = 4 - 4 \times 1 \times (-15) = 64 > 0$

$\therefore$ $x^2 = 2x + 15$ 有两个不相等的实数根.

$$x = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2}$$

$$x_1 = 1 + 4 = 5, \qquad x_2 = 1 - 4 = -3$$

Example 4.17 $x^2 = 2x + 15$

Solution $x^2 - 2x - 15 = 0$ Put in standard form.

$$a = 1, \quad b = -2, \quad c = -15$$

$$b^2 - 4ac = 4 - 4 \times 1 \times (-15) = 64 > 0$$

Thus, there are two unequal real roots for equation $x^2 = 2x + 15$.

$$x = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2}$$

$$x_1 = 1 + 4 = 5, \qquad x_2 = 1 - 4 = -3$$

例 4.18 $4x^2 - 4x + 9 = 0$

解: $a = 4, \quad b = -4, \quad c = 9$

$$b^2 - 4ac = 16 - 4 \times 4 \times 9 = 16 - 144$$

$$= -128 < 0$$

$\therefore$ 方程 $4x^2 - 4x + 9 = 0$ 没有实数根.

Example 4.18 $4x^2 - 4x + 9 = 0$

Solution $a = 4 \quad b = -4 \quad c = 9$

$$b^2 - 4ac = 16 - 4 \times 4 \times 9 = 16 - 144$$

$$= -128 < 0$$

Thus there are no real solutions for equation $4x^2 - 4x + 9 = 0$.

例 4.19 $9x^2 + 6x + 1 = 0$

解: $a = 9, \quad b = 6, \quad c = 1$

$$b^2 - 4ac = 36 - 4 \times 9 \times 1 = 0$$

$\therefore$ 方程 $9x^2 - 6x + 1 = 0$ 有两个相等的实数根.

Example 4.19 $9x^2 + 6x + 1 = 0$

Solution $a = 9, \quad b = 6, \quad c = 1$

$b^2 - 4ac = 36 - 4 \times 9 \times 1 = 0$

Hence there are two equal real roots for equation $9x^2 - 6x + 1 = 0$.

三、线性方程组的求解(Solution for Systems of Linear Equations)

(一) 二元线性方程组的求解(Solution for Systems of Linear Equations with Two Variables)

1. 加减消元法(Solution by the Addition Method)

求解步骤:

① 一个方程或两个方程两边同乘相应的常数,以使其中一个变量在两个方程中的系数相同或互为反数;

② 将两个方程相加减,以得到只有一个未知数的新方程;

③ 从新方程中解出第一个未知数的值;

④ 将第一未知数的解代入原方程中任一个求解第二个变量;

⑤ 将两个变量的解分别代入原方程检验.

Procedures for solution:

① Multiply both sides of one or two equations by a corresponding constant so that the coefficients of one of the variables are similar or inverse to each other;

② Add corresponding members of the equations to obtain a new equation containing a single variable;

③ Solve the derived equation for that variable;

④ Substitute the value of the one variable into either of the original equations to solve for the second variable;

⑤ State the solution after checking it in the original system.

例 4.20 解方程

$$3x - 2y = 15$$

$$x + y = 10$$

$3x - 2y = 15$

$+ 2x + 2y = 20$　　第二个方程乘 2.

$5x \quad = 35$　　将两个方程相加.

$x = 7$　　解 x.

$(7) + y = 10.$　　将 $x = 7$ 代入原方程组第二个方程.

$y = 3$　　解 y.

检验：

$3(7) - 2(3) = 15$　　检验第一个方程.

$21 - 6 = 15$

$7 + 3 = 10$　　检验第二个方程.

∴　解为　$x = 7$

$y = 3$

Example 4.20　Solve for the equations

$$\begin{cases} 3x - 2y = 15 \\ x + y = 10 \end{cases}$$

$3x - 2y = 15$

$+ 2x + 2y = 20$　　Multiply the second equation by 2.

$5x \quad = 35$　　Add the two equations.

$x = 7$　　Solve for x .

$(7) + y = 10$　　Substitute the numerical value of x into either of the original equations.

$y = 3$　　Solve for y .

Check:

$3(7) - 2(3) = 15$　　Check the first equation.

$21 - 6 = 15$

$7 + 3 = 10$　　Check the second equation.

Therefore, the solutions are

$x = 7, \quad y = 3$

例 4.21　解方程 $\begin{cases} 3x + 2y = 15 \\ 5x - 3y = 6 \end{cases}$

$$\begin{aligned} 9x + 6y &= 45 && \text{第一个方程乘 3.} \\ +\ 10x - 6y &= 12 && \text{第二个方程乘 2.} \\ 19x &= 57 && \text{将两个方程相加.} \\ x &= 3 && \text{求解 } x. \\ 3(3) + 2y &= 15 && \text{将 } x = 3 \text{ 代入原方程组第一个方程求 } y. \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

检验：$3(3) + 2(3) = 15$　检验第一个方程.

$9 + 6 = 15$

$5(3) - 3(3) = 6$　检验第二个方程.

$15 - 9 = 6$

$\therefore \quad x = 3, \quad y = 3$

Example 4.21　Solve the equations:

$$\begin{cases} 3x + 2y = 15 \\ 5x - 3y = 6 \end{cases}$$

$$\begin{aligned} 9x + 6y &= 45 && \text{Multiply the first equation by 3.} \\ +\ 10x - 6y &= 12 && \text{Multiply the second equation by 2.} \\ 19x &= 57 && \text{Add the two equations.} \\ x &= 3 && \text{Solve for } x. \\ 3(3) + 2y &= 15 && \text{Substitute to find } y. \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

Check: $3(3) + 2(3) = 15$　Check the first equation.

$9 + 6 = 15$

$5(3) - 3(3) = 6$ Check the second equation.

$15 - 9 = 6$

Therefore, $x = 3$, $y = 3$

2. 代入消元法(Substitution Method)

求解步骤:

① 将其中一个方程写成一个变量为另一个变量的等代关系;

② 将所得关系式代入另一个方程;

③ 求解只有一个变量的方程,以得到该变量的值;

④ 将该变量的值代入任一个原方程以求出另一变量的值;

⑤ 将所求得的解代入原方程进行检验.

Procedures for the solution:

① Solve one of the equations for one of the variables;

② Substitute the expression that you obtain into the other equation;

③ Solve the resulting equation in a single variable for the value of that variable;

④ Substitute that value into either of the original equations to determine the value of the other variable;

⑤ State the solution after checking it in the original system.

例 4.22 解方程组

$$\begin{cases} 3x - 2y = 15 \\ x + y = 10 \end{cases}$$

解: 用第二个方程

$x + y = 10$

$y = -x + 10$ 找出 y 的表达式.

$3x - 2(-x + 10) = 15$ 将 y 的表达式代入第一个方程.

$3x + 2x - 20 = 15$

$5x = 35$ 解出 x.

$x = 7.$

$7 + y = 10$ 将 x 值代入原方程组第二个方程求出 y.

$y = 3$

∴ 解为 $x = 7, \quad y = 3$

Example 4.22 Solve for the system of equations

$$\begin{cases} 3x - 2y = 15 \\ x + y = 10 \end{cases}$$

Solution Use the second equation.

$x + y = 10$

$y = -x + 10$ Find an expression for y.

$3x - 2(-x + 10) = 15$ Substitute into the first equation.

$3x + 2x - 20 = 15$

$5x = 35$ Solve for x.

$x = 7$

$7 + y = 10$ Substitute the value of x into the second

$y = 3$ original equation to find y.

Therefore, the solutions are

$x = 7, \quad y = 3$

例 4.23 解方程组

$$\begin{cases} 3x + 2y = 15 \\ 5x - 3y = 6 \end{cases}$$

解： 用第二个方程

$5x - 3y = 6$

$3y = 5x - 6$ 找出 y 的表达式.

$$y = \frac{5x - 6}{3}$$

$3x + 2\left(\frac{5x - 6}{3}\right) = 15$ 把 y 的表达式代入第一个方程.

$3x + \frac{10x - 12}{3} = 15$ 求解 x.

$9x + 10x - 12 = 45$

$19x - 12 = 45$

$19x = 57$

$x = 3$

$5(3) - 3y = 6$ 代入原方程 2,求解 y.

$15 - 3y = 6$

$3y = 9$

$y = 3$

∴ 方程组的解为 $x = 3,\quad y = 3$

Example 4.23 Solve for the system of equations

$$\begin{cases} 3x + 2y = 15 \\ 5x - 3y = 6 \end{cases}$$

Solution Use the second equation:

$5x - 3y = 6$

$3y = 5x - 6$ Find an expression for y.

$y = \frac{5x - 6}{3}$

$3x + 2\left(\frac{5x - 6}{3}\right) = 15$ Substitute into the first equation.

$3x + \frac{10x - 12}{3} = 15$ Solve for x.

$9x + 10x - 12 = 45$

$19x - 12 = 45$

$19x = 57$

$x = 3$

$5(3) - 3y = 6$ Substitute into the second

original equation to find y.

$$15 - 3y = 6$$
$$3y = 9$$
$$y = 3$$

Therefore, the solutions are $x = 3, \quad y = 3$

2. 三元线性方程组求解

(Solve for Systems of Linear Equations wih Three Variables)

三元线性方程的求解大多也采用加减消元法和代入消元法，其原理和基本步骤与二元线性方程组的求解相同.

The solution of a system of linear equations with three variables is also done by using the methods of elimination and substitution. The principle and procedure are the same with a system of linear equations with two variables.

例 4.24 用代入法和加减消元法求解下列方程组

$$\begin{cases} 3x + 3y + z = 17 & ① \\ 3x + y + 3z = 15 & ② \\ x + 3y + 3z = 13 & ③ \end{cases}$$

解 1: 用代入法求解.

由方程 ①,我们得

$$z = 17 - 3x - 3y \quad ④$$

将方程 ④代入②,③,得

$$3x + y + 3(17 - 3x - 3y) = 15 \quad ⑤$$
$$x + 3y + 3(17 - 3x - 3y) = 13 \quad ⑥$$

重排方程⑤,⑥,得

$$-6x - 8y = -36$$
$$-8x - 6y = -38$$

再用求解二元线性方程的方法求出 x, y, 最终求得 z.

由此可得 $x = 3\frac{1}{7}$, $y = 2\frac{1}{7}$, $z = 1\frac{1}{7}$

Example 4.24 Use the methods of substitution and elimination to find the solution of the system

$$\begin{cases} 3x + 3y + z = 17 & ① \\ 3x + y + 3z = 15 & ② \\ x + 3y + 3z = 13 & ③ \end{cases}$$

Solution 1 Use the method of substitution to find the solution.

From ①, get

$$z = 17 - 3x - 3y \quad ④$$

Bring ④to ② and ③ to get

$$3x + y + 3(17 - 3x - 3y) = 15 \quad ⑤$$

$$x + 3y + 3(17 - 3x - 3y) = 13 \quad ⑥$$

Reorder ⑤ and ⑥ to get

$$-6x - 8y = -36$$

$$-8x - 6y = -38$$

Then use the method of solving linear equations to get x and y, and in the end to get z.

From which we obtain $x = 3\frac{1}{7}$, $y = 2\frac{1}{7}$, $z = 1\frac{1}{7}$

解 2: 用加减消元法求解上述方程组

$$\begin{cases} 3x + 3y + z = 17 & ① \\ 3x + y + 3z = 15 & ② \\ x + 3y + 3z = 13 & ③ \end{cases}$$

将方程 ①乘以 3, 得

$$9x + 9y + 3z = 51 \quad ④$$

用方程④减方程②,得

$$6x + 8y = 36 \quad ⑤$$

用方程 ② 减方程③,得

$$2x - 2y = 2 \quad ⑥$$

由方程⑤和⑥,得出 x, y,最终再求得 z,由此得

$$x = 3\frac{1}{7}, \quad y = 2\frac{1}{7}, \quad z = 1\frac{1}{7}$$

Solution 2 Use the method of elimination to solve the above system of equations

$$\begin{cases} 3x + 3y + z = 17 & ① \\ 3x + y + 3z = 15 & ② \\ x + 3y + 3z = 13 & ③ \end{cases}$$

Multiply equation ① by 3 and get

$$9x + 9y + 3z = 51 \quad ④$$

Subtract equation ④ and equation ②, and get

$$6x + 8y = 36 \quad ⑤$$

Subtract equation ② and equation ③, and get

$$2x - 2y = 2 \quad ⑥$$

From equations ⑤ and ⑥, obtain x and y, and in the end get

$$x = 3\frac{1}{7}, \quad y = 2\frac{1}{7}, \quad z = 1\frac{1}{7}$$

§4.2 不等式的表达(Inequalities)

一、概述(Introduction)

(一) 定义(Definitions)

用关系符号 $<$, $\leq$, $>$ 和 $\geq$ 联结的表达式称为不等式.若不等式中含有1个关系符号,称为单不等式,例:$4 < 7$, $3x > 10$;若不等式中含有2个以上关系符号,则称为复不等式,例 $x < y \leq z$,

$5 \geq 2x - 3 \geq 1$ 等都称为复不等式.一个复不等式,能拆成若干个单不等式, 例:$x < y \leq z$ 可拆成 $x < y$, $y \leq z$ 等.

The expressions which are connected by order symbols $<$, $\leq$, $>$ and $\geq$ are called inequalities. If there is one order symbol in an inequality, the inequality is called a single inequality, such as $4 < 7$, $3x > 10$; if there are two or more than two order symbols in an inequality, we call it a complex inequality, such as $x < y \leq z$, $5 \geq 2x - 3 \geq 1$. A complex inequality can always be broken down into single inequalities: for instance, $x < y \leq z$ means $x < y$ and $y \leq z$.

(二) 不等式的性质(Properties of Inequalities)

设 a, b, c 为实数.

Let a, b and c denote real numbers.

1. 如果 $a < b$, $b < c$, 则 $a < c$.

If $a < b$ and $b < c$, then $a < c$.

2. 如果 $a < b$,则 $a + c < b + c$, $a - c < b - c$.

If $a < b$, then $a + c < b + c$, $a - c < b - c$.

3. 如果 $a < b$, $c > 0$, 则 $ac < bc$; $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c > 0$, then $ac < bc$; $\frac{a}{c} < \frac{b}{c}$.

4. 如果 $a < b$, $c < 0$, 则 $ac > bc$; $\frac{a}{c} > \frac{b}{c}$.

If $a < b$ and $c < 0$, then $ac > bc$; $\frac{a}{c} > \frac{b}{c}$.

例 4.25 解下列不等式.

(1) $x + 1 > 3$ 不等式两边同减 1.

$$x > 2$$

(2) $x - 2 < 3$ 不等式两边同加 2.

$$x < 5$$

(3) $3x \leq 6$ 不等式两边同除 3.

$$x \leq 2$$

(4) $-0.2x \leq -1$ 不等式两边同除 -0.2.

$$x \geq 5$$

Example 4.25 Solve the following inequalities.

(1) $x + 1 > 3$ Subtract 1 from both sides.

$$x > 2$$

(2) $x - 2 < 3$ Add 2 to both sides.

$$x < 5$$

(3) $3x \leq 6$ Divide both sides by 3.

$$x \leq 2$$

(4) $-0.2x \leq -1$ Divide both sides by -0.2.

$$x \geq 5$$

二、一元一次不等式(First-degree Inequalities with One Variable)

不等式中含有一个未知数,且未知数的次数都是 1 的不等式叫做一元一次不等式.满足不等式条件的未知数的解称为不等式的解,它通常指数轴上的一定区间的数.

An inequality containing one unknown whose degree is 1 is called a first-degree inequality with one variable. The solution for the unknown to satisfy the inequality is called the solution set to the inequality, and it is usually a series of numbers in the interval of a number line.

解不等式的方法与解方程的方法相同,只是当不等式的各项乘以或除以一个负值时,必须改变不等式的所有符号. 通常的方

法是将未知数移到不等式的一边.

To solve an inequality, that is , to find its solution set, you proceed in much the same way as in solving equations, except of course that when you multiply or divide all members by a negative quantity, you must reverse all signs of inequality. The usual approach is to try to isolate the unknown on one side of the inequality.

例 4.26　解不等式 $5x + 3 < 18$,并将其解集标在数轴上.

解：　$5x + 3 < 18$

$5x < 15$　　两边分别减去 3.

$x < 3$　　两边分别除以 5.

Example 4.26　Solve the inequality $5x + 3 < 18$ and sketch its solution set on a number line.

Solution　$5x + 3 < 18$

$5x < 15$　　Subtract 3 from both sides.

$x < 3$　　Divide both sides by 5.

由于最后一个不等式与原不等式等值,所以我们得出结论,$5x + 3 < 18$ 的解是由 $x < 3$ 范围内的所有 x 数组成,在数轴上的区间为$(-\infty, 3)$(见图 4.1).

Since the last inequality is equivalent to the original one, we conclude that the solution of $5x + 3 < 18$ consists of all numbers x such that $x < 3$, whose interval on a number line is $(-\infty, 3)$(see figure 4.1).

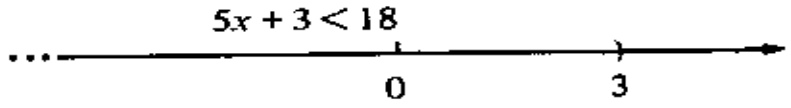


图 4.1(Figure 4.1)

例 4.27　解不等式 $3-2x \geq 4x-9$，并将其解集标在数轴上.

解：　$3-2x \geq 4x-9$

$3-6x \geq -9$　　两边分别减去 $4x$.

$-6x \geq -12$　　两边分别减去 3.

$x \leq 2$　　两边分别除以 -6(改变符号).

在数轴上的区间为$(-\infty, 2]$(见图 4.2).

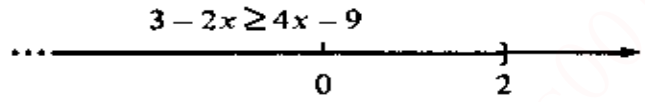


图 4.2(Figure 4.2)

Example 4.27　Solve the inequality $3-2x \geq 4x-9$ and sketch its solution set on a number line.

Solution　$3-2x \geq 4x-9$

$3-6x \geq -9$　　Subtract $4x$ from both sides.

$-6x \geq -12$　　Subtract 3 from both sides.

$x \leq 2$　　Divide both sides by -6 (reverse the inequality).

Therefore, the solution set is $(-\infty, 2]$(see figure 4.2).

例 4.28　解不等式 $\frac{5}{6}x-\frac{3}{4} \geq \frac{1}{2}x-\frac{2}{3}$，将解集标在数轴上.

解：　我们先将两边乘以分数系数的最小公分母 12

$10x-9 \geq 6x-8$

$4x-9 \geq -8$　　两边分别减去 $6x$.

$4x \geq 1$　　两边分别加上 9.

$x \geq \frac{1}{4}$　　两边分别除以 4.

由此得不等式的解集为$[\frac{1}{4},\infty)$(见图 4.3).

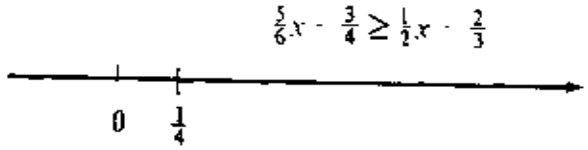


图 4.3(Figure 4.3)

Example 4.28 Solve the inequality $\frac{5}{6}x-\frac{3}{4}\geq\frac{1}{2}x-\frac{2}{3}$ and sketch its solution set on a number line.

Solution We begin by multiplying both sides by 12, the LCD (least common denominator) of the fractional coefficients:

$10x-9\geq 6x-8$

$4x-9\geq -8$ Sobtract $6x$ from both sides.

$4x\geq 1$ Add 9 to both sides.

$x\geq\frac{1}{4}$ Divide both sides by 4.

Therefore, the solution set is $[\frac{1}{4},\infty)$(see figure 4.3).

例 4.29 $-11\leq 2x-3<7$

解: 我们先在各项上分别加上 3,使 x 处于中间项

$-8\leq 2x<10$

$-4\leq x<5$ 各项除以 2.

由此得解集为$[-4,5)$(见图 4.4).

Example 4.29 $-11\leq 2x-3<7$

Solution We begin by adding 3 to all terms to help isolate x in the middle:

$-8\leq 2x<10$

$-4 \leq x < 5$ Divide all terms by 2.

Therefore, the solution set is $[-4,5)$ (see figure 4.4).

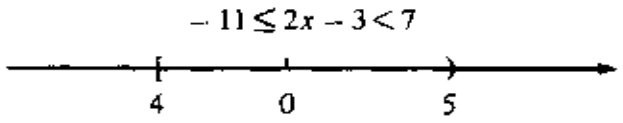


图 4.4(Figure 4.4)

三、绝对值不等式的求解

(Solution for Inequalities with Absolute Values)

带有绝对值的不等式称为绝对值不等式.

Inequalities involving absolute values are called the inequalities with absolute values.

(一) 不等式 $|u| < a$ 和 $|u| \leq a$ 的求解.

(Solution for the Inequalities $|u| < a$ and $|u| \leq a$)

如果 $a > 0$,则:

(1) $|u| < a$, 条件为 $-a < u < a$.

(2) $|u| \leq a$, 条件为 $-a \leq u \leq a$.

If $a > 0$, then:

(1) $|u| < a$, if and only if $-a < u < a$.

(2) $|u| \leq a$, if and only if $-a \leq u \leq a$.

例 4.30 解不等式 $|2x+3| < 5$

解: $|2x+3| < 5$

$-5 < 2x+3 < 5$

$-8 < 2x < 2$ 各项分别减去 3.

$-4 < x < 1$ 各项除以 2.

由此得不等式的解集为 $(-4,1)$ (见图 4.5).

Example 4.30 Solve the inequality $|2x + 3| < 5$.

Solution $|2x + 3| < 5$

$-5 < 2x + 3 < 5$

$-8 < 2x < 2$ Subtract 3 from all members.

$-4 < x < 1$ Divide all members by 2.

Hence, the solution set is the interval $(-4,1)$ (see figure 4.5).

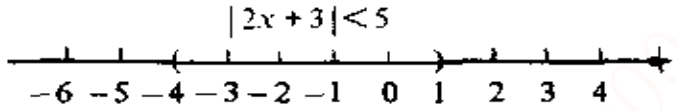


图 4.5(Figure 4.5)

例 4.31 解不等式 $|2x| + 1 \leq 5$

解: $|2x| + 1 \leq 5$

$|2x| \leq 4$ 两边减去 1.

$-4 \leq 2x \leq 4$

$-2 \leq x \leq 2$ 各项除以 2.

由此得不等式的解集为$[-2,2]$(见图 4.6).

Example 4.31 Solve the inequality $|2x| + 1 \leq 5$.

Solution $|2x| + 1 \leq 5$

$|2x| \leq 4$ Subtract 1 from both sides.

$-4 \leq 2x \leq 4$

$-2 \leq x \leq 2$ Divide all members by 2.

Hence, the solution set is the interval $[-2,2]$ (see figure 4.6).

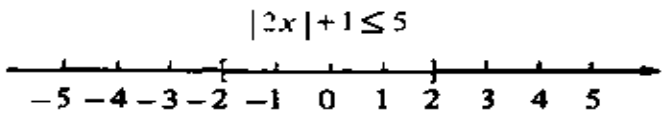


图 4.6(Figure 4.6)

(二) $|u| > a$, $|u| \geq a$ 的求解.

(Solution for the Inequalities $|u| > a$, $|u| \geq a$)

如果 $a > 0$, 那么:

1. $|u| > a$, 条件为 $u < -a$, 或 $u > a$.
2. $|u| \geq a$, 条件为 $u \leq -a$, 或 $u \geq a$.

If $a > 0$, then:

1. $|u| > a$ if and only if $u < -a$ or $u > a$.
2. $|u| \geq a$ if and only if $u \leq -a$ or $u \geq a$.

例 4.32 解不等式 $\left|\frac{1}{2}x - 1\right| \geq 2$, 并在数轴上标出解集.

解: $\left|\frac{1}{2}x - 1\right| \geq 2$

$$\frac{1}{2}x - 1 \leq -2 \quad 或 \quad \frac{1}{2}x - 1 \geq 2$$

$$\frac{1}{2}x \leq -1 \quad 或 \quad \frac{1}{2}x \geq 3$$

$$x \leq -2 \quad 或 \quad x \geq 6$$

因此,该不等式的解集由区间$(-\infty, -2]$或区间$[6, \infty)$上的所有值组成(见图 4.7).

Example 4.32 Solve the inequality $\left|\frac{1}{2}x - 1\right| \geq 2$ and sketch its solution set on a number line.

Solution $\left|\frac{1}{2}x - 1\right| \geq 2$

$$\frac{1}{2}x - 1 \leq -2 \quad \text{or} \quad \frac{1}{2}x - 1 \geq 2$$

$$\frac{1}{2}x \leq -1 \quad \text{or} \quad \frac{1}{2}x \geq 3$$

$$x \leq -2 \quad \text{or} \quad x \geq 6$$

Thus, the solution set consists of all values that belong to either of

the intervals $(-\infty, -2]$ or $[6, \infty)$(see figure 4.7).

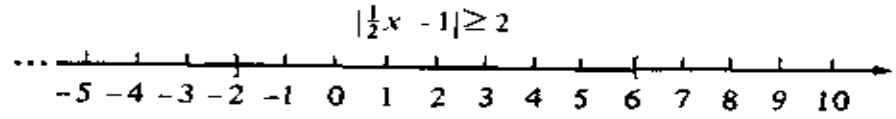


图 4.7(Figure 4.7)

四、多次不等式(Polynomial Inequalities)

不等号两边未知数的次数高于 2 次的不等式称为多次不等式.例:

$$4x^3 + 3x^2 + x - 1 > x^2 + 3x - x$$

The inequality in which the order of the unknown on both sides is greater than 2 is called a polynomial inequality. For example,

$$4x^3 + 3x^2 + x - 1 > x^2 + 3x - x.$$

多次不等式的求解步骤如下:

① 化多次不等式为标准多次不等式,即不等号的右边为 0,左边为多项式;

② 将不等式右边多项式等于 0,并解出其所有实根;

③ 将第②步获得的根按序排列在数轴上.这些根将数轴分成若干个开区域;

④ 多项式的代数符号在上一步得到的任意区域内都不会改变.选择一个测试数字来决定每个区域的符号,并将该测试数字代入多项式的未知数,所得到的值的代数符号即为这个多项式在整个区域上的符号;

⑤ 作图标出第④步中该多项式在不同开区间上的代数符号.不等式的解集均可在图上读出.

Procedures for solving polynomial inequalities:

① Change an inequality into its standard form, with zero on the right side and a polynomial on the left side;

② Let the polynomial on the right of the inequality equal to zero and find all real roots of the resulting equation;

③ Arrange the roots obtained in step ② in increasing order on a number line and these roots will divide the number line into several open intervals;

④ The algebraic sign of the polynomial cannot change over any of the intervals obtained in step ③. Determine the sign for each interval by selecting a test number in the interval and substitute it for the unknown in the polynomial. The algebraic sign of the resulting value is the sign of the polynomial over the entire interval;

⑤ Draw a figure showing the information obtained in step ④ about the algebraic sign of the polynomial over the various open intervals and the solution set of the inequality can be read from this figure.

例 4.33　解　$4x^2 + 8x \geq 5$

解：　按以上的步骤求解：

① 将不等式变为　$4x^2 + 8x - 5 \geq 0$

② 写出 $4x^2 + 8x - 5 = 0$，因式分解左边得

$$(2x + 5)(2x - 1) = 0$$

根为　$x_1 = -\frac{5}{2}$，　$x_2 = \frac{1}{2}$

③ $4x^2 + 8x - 5$ 的代数符号为常数的开区间为$(-\infty, -\frac{5}{2})$，$(-\frac{5}{2}, \frac{1}{2})$，$(\frac{1}{2}, \infty)$(见图 4.8).

④ 对于区间$(-\infty, -\frac{5}{2})$，我们在多项式$4x^2 + 8x - 5$中代入

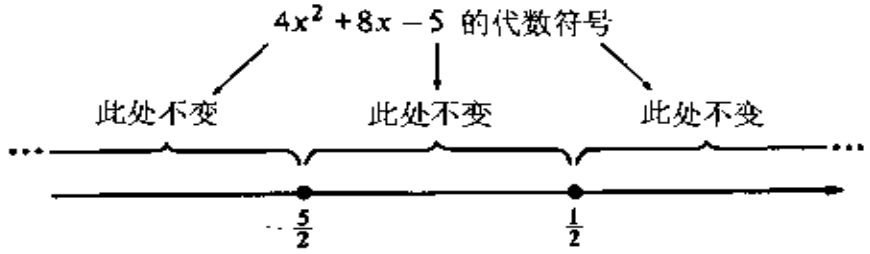


图 4.8

测试数如 $x = -3$,得:

$$4(-3)^2 + 8(-3) - 5 = 7 > 0$$

我们得到 $4x^2 + 8x - 5 > 0$ 中的所有 x 值均在 $(-\infty, \frac{5}{2})$ 区间内. 对于区间 $(-\frac{5}{2}, \frac{1}{2})$,代入测试数如 $x = 0$,得:

$$4(0)^2 + 8(0) - 5 = -5 < 0$$

因此,$4x^2 + 8x - 5 < 0$ 中的所有 x 值均在 $(-\frac{5}{2}, \frac{1}{2})$ 区间内. 对于区间 $(\frac{1}{2}, \infty)$,代入测试数 $x = 1$,得:

$$4(1)^2 + 8(1) - 5 = 7 > 0$$

因此,$4x^2 + 8x - 5 < 0$ 中的所有 x 值均在区间 $(\frac{1}{2}, \infty)$.

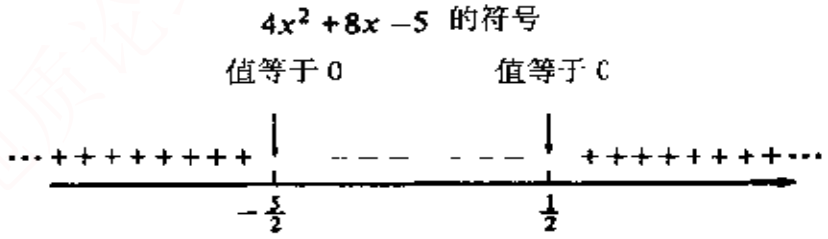


图 4.9

⑤ 第 ④ 步得到的信息可概括为图 4.9. 从图中可以看到 $4x^2 + 8x - 5 \geq 0$ 的解为 $x \leq -\frac{5}{2}$ 和 $x \geq \frac{1}{2}$. 换言之,解集

由所有的实数组成，这些实数或属于区间$(-\infty, -\frac{5}{2}]$，或属于区间$[\frac{1}{2}, \infty)$. 因此，不等式的解集为$(-\infty, -\frac{5}{2}] \cup [\frac{1}{2}, \infty)$.

Example 4.33 Solve for $4x^2 + 8x \geq 5$

Solution Carry out the steps in the procedures above.

① The given inequality is equivalent to $4x^2 + 8x - 5 \geq 0$.

② Write $4x^2 + 8x - 5 = 0$ and factor the left side to obtain

$$(2x + 5)(2x - 1) = 0.$$

The roots are $x_1 = -\frac{5}{2}$ and $x_2 = \frac{1}{2}$.

③ The open intervals over which the algebraic sign of $4x^2 + 8x - 5$ is constant are $(-\infty, -\frac{5}{2})$, $(-\frac{5}{2}, \frac{1}{2})$, and $(\frac{1}{2}, \infty)$ (see figure 4.8).

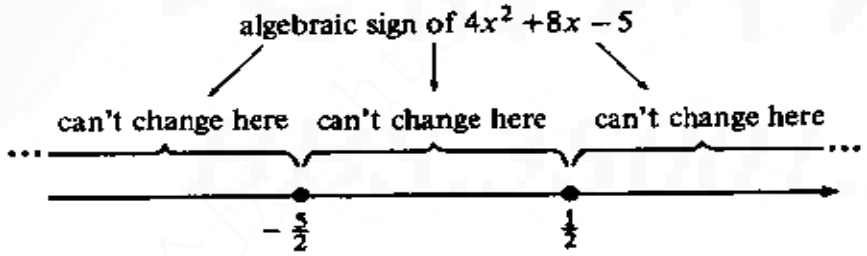


Figure 4.8

④ For the interval $(-\infty, -\frac{5}{2})$, we substitute a test number, say, $x = -3$ in the polynomial $4x^2 + 8x - 5$ to obtain

$$4(-3)^2 + 8(-3) - 5 = 7 > 0.$$

We conclude that $4x^2 + 8x - 5 > 0$ for all values of x in the

interval $(-\infty, \frac{5}{2})$. For the interval $(-\frac{5}{2}, \frac{1}{2})$, we substitute a test number, say, $x = 0$, in the polynomial to get

$$4(0)^2 + 8(0) - 5 = -5 < 0.$$

Therefore, $4x^2 + 8x - 5 < 0$ for all values of x in the interval $(-\frac{5}{2}, \frac{1}{2})$. For the interval $(\frac{1}{2}, \infty)$, we substitute a test number, say, $x = 1$, and find that

$$4(1)^2 + 8(1) - 5 = 7 > 0.$$

Hence, $4x^2 + 8x - 5 < 0$ for all values of x in the interval $(\frac{1}{2}, \infty)$.

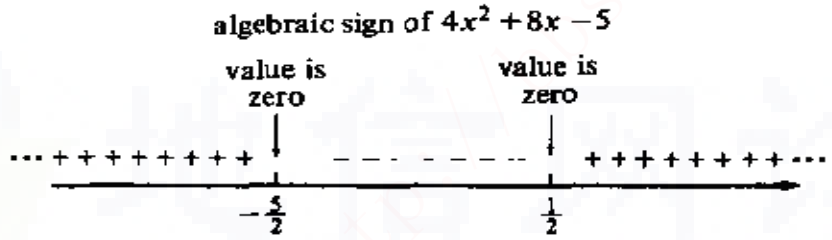


Figure 4.9

⑤ The information obtained in step 4 is summarized in figure 4.9, from which we see that $4x^2 + 8x - 5 \geq 0$ for $x \leq -\frac{5}{2}$ and also for $x \geq \frac{1}{2}$. In other words, the solution set consists of all real numbers thet belong to either interval $(-\infty, -\frac{5}{2}]$ or interval $[\frac{1}{2}, \infty)$. Thus, the solution set is $(-\infty, -\frac{5}{2}] \cup [\frac{1}{2}, \infty)$.

第五章　矩阵和行列式的表达
(Matrices and Determinants)

§5.1　矩阵的表达(Matrices)

一、定义(Definitions)

矩阵是 $m \times n$ 个数排列成 m 行 n 列的长方形阵列，用记号

$$\mathbf{A}_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} = [a_{ij}]$$

来表示．一般地，记为 $\mathbf{A}_{m\times n} = [a_{ij}]$，$a_{ij}$ 称为矩阵的元素．第一个下标指行号，第二个下标指列号．矩阵的元素可以是实数，或代数式．$\mathbf{A}_{m\times n}$ 又称为 $m \times n$ 阶矩阵，当 $m = n$ 时，称为方阵．而 $m \times 1$ 阶矩阵称为列矩阵，$1 \times n$ 阶矩阵称为行矩阵．

A matrix is a rectangular array of entries with dimension $m \times n$, where m is the number of rows and n is the number of columns of the entries. A matrix can be symbolized as

$$\mathbf{A}_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} = [a_{ij}]$$

In general, it is written as $\mathbf{A}_{m\times n} = [a_{ij}]$ in which a_{ij} is called the element of the matrix. The first subscript indicates the row, and the second subscript indicates the column. The entries of a matrix may be real numbers or algebraic expressions. $\mathbf{A}^{m\times n}$ is also called a $m \times n$ matrix. If $m = n$, the matrix is called a square matrix. $m \times 1$ matrix is called a column matrix. $1 \times n$ matrix is called a row matrix.

例 5.1(Example 5.1)

$$\mathbf{A}_{2\times 3} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

为 2×3 阶矩阵.
is 2×3 matrix.

$$\mathbf{B}_{2\times 2} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

为 2×2 阶方矩阵.
is 2×2 square matrix.

$$\mathbf{C}_{3\times 1} = \begin{bmatrix} 7 \\ -9 \\ 4 \end{bmatrix}$$

为 3×1 阶列矩阵.
is 3×1 column matrix.

二、矩阵的定理(Properties of Matrices)

(一) 矩阵的相等(Equality of Matrices)

如果两个矩阵 $\mathbf{A}_{m\times n} = [a_{ij}]$ 和 $\mathbf{B}_{m\times n} = [b_{ij}]$,则 $\mathbf{A}_{m\times n} = \mathbf{B}_{m\times n}$.这意味着当也只有当两个矩阵的大小和所有对应元素都相等时,两个矩阵才相等.

If two matrices $\mathbf{A}_{m\times n} = [a_{ij}]$ and $\mathbf{B}_{m\times n} = [b_{ij}]$, then $\mathbf{A}_{m\times n} = \mathbf{B}_{m\times n}$. This means that two matrices are equal if and only if they are of the same dimension and all corresponding entries are equal.

(二) 矩阵的加法(Matrix Addition)

如果两个矩阵具有相同的行与列时,才可进行相加.两矩阵相加就是将两矩阵的对应元素相加.这可由下列定义来描述:

如果 $\mathbf{A}_{m\times n} = [a_{ij}]$, $\mathbf{B}_{m\times n} = [b_{ij}]$, 则 $\mathbf{A} + \mathbf{B} = [a_{ij}b_{ij}]_{m\times n}$

If two matrices have the same dimension, they are said to be conformable for addition. The matrices are added by adding corresponding entries of the given matrices. This is described formally in the following definition:

If $\mathbf{A}_{m\times n} = [a_{ij}]$, $\mathbf{B}_{m\times n} = [b_{ij}]$, then $\mathbf{A} + \mathbf{B} = [a_{ij}b_{ij}]_{m\times n}$

例 5.2(Example 5.2)

已知(Given) $\mathbf{A} = [4 \quad 2 \quad 6]$ $\mathbf{B} = [1 \quad 7 \quad 2]$

$$\mathbf{C} = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 7 & 3 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 6 & 1 & 2 \\ 3 & -10 & 4 \\ 1 & 3 & -2 \end{bmatrix}$$

解(Solution): 将各矩阵相加,得

Add the matrices to get

$\mathbf{A} + \mathbf{B} = [4+1 \quad 2+7 \quad 6+2] = [5 \quad 9 \quad 8]$

$\mathbf{A} + \mathbf{C}$ 1×3 与 2×3 不可加.

1×3 and 2×3 may not be added.

$\mathbf{C} + \mathbf{E}$ 2×3 与 3×3 不可加.

2×3 and 3×3 may not be added.

$$\mathbf{D} + \mathbf{E} = \begin{bmatrix} 2+6 & 1+1 & 0+2 \\ 4+3 & 7+(-10) & 3+4 \\ -2+1 & 0+3 & 1+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 2 \\ 7 & -3 & 7 \\ -1 & 3 & -1 \end{bmatrix}$$

(三) 矩阵与实数的乘积(Scalar Multiplication)

矩阵与实数相乘,就是矩阵中的每一元素与实数相乘,即:如果 $\mathbf{A}^{m\times n}=[a_{ij}]$,那么,对于任意实数 c,有 $c\mathbf{A}=[ca_{ij}]$.

This operation of multiplying each entry in a matrix by a number is called scalar multiplication, that is, if $\mathbf{A}_{m\times n}=[a_{ij}]$, then for any real number c, $c\mathbf{A}=[ca_{ij}]$.

例 5.3(Example 5.3)

已知(Given) $\mathbf{A}=\begin{bmatrix}2&1&0\\-2&0&1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}6&1&2\\1&3&-2\end{bmatrix}$

求(Find) $3\mathbf{A}+2\mathbf{B}$

解(Solution):

$$3\mathbf{A}+2\mathbf{B}=\begin{bmatrix}3\cdot2&3\cdot1&3\cdot0\\3\cdot(-2)&3\cdot0&3\cdot1\end{bmatrix}+\begin{bmatrix}2\cdot6&2\cdot1&2\cdot2\\2\cdot1&2\cdot3&2\cdot(-2)\end{bmatrix}$$

$$=\begin{bmatrix}18&5&4\\-4&6&-1\end{bmatrix}$$

(四) 矩阵的相乘(Matrix Multiplication)

设有 $\mathbf{A}_{m\times n}$ 和 $\mathbf{B}_{n\times p}$ 两个矩阵,且 $\mathbf{A}$ 矩阵的列数与 $\mathbf{B}$ 矩阵的行数相等,则有:

$$\mathbf{AB}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{i1}&a_{i2}&\cdots&a_{in}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1j}&\cdots&b_{1p}\\b_{21}&b_{22}&\cdots&b_{2j}&\cdots&b_{2p}\\\vdots&\vdots&&\vdots&&\vdots\\b_{n1}&b_{n2}&\cdots&b_{nj}&\cdots&b_{np}\end{bmatrix}$$

$$= [c_{ij}] = \mathbf{C}_{m\times p}$$

c_{ij} 为 $m \times p$ 阶矩阵 $\mathbf{C}_{m\times p}$ 中的一个元素

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

If there are two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times p}$ and the columns of matrix **A** are equal to the rows of matrix **B**, then

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

$$= [c_{ij}] = C_{m\times p}$$

where the element c_{ij} of the product $\mathbf{C}_{m\times p}$ is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

例 5.4(Example 5.4)

求(Find)**AB**,设(let)$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 4 & -1 & 2 & 6 \\ 1 & 3 & -1 & 2 \end{bmatrix}$

解(Solution):

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 6 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 4 + 2\cdot 1 & 1(-1) + 2\cdot 3 & 1\cdot 2 + 2(-1) & 1\cdot 6 + 2\cdot 2 \\ 3\cdot 4 + 4\cdot 1 & 3(-1) + 4\cdot 3 & 3\cdot 2 + 4(-1) & 3\cdot 6 + 4\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 0 & 10 \\ 16 & 9 & 2 & 26 \end{bmatrix}$$

例 5.5(Example 5.5)

已知(Given) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

$$\mathbf{C}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} \qquad \mathbf{D}=\begin{bmatrix}1 & 0 & 2 & 1\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\end{bmatrix}$$

求解(Find)： **AB**, **BA**, **CD**

解(Solution)：

$$\mathbf{AB}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1\cdot(-1)+2\cdot 1 & 1\cdot 2+2\cdot 3\\ 3\cdot(-1)+4\cdot 1 & 3\cdot 2+4\cdot 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 8\\ 1 & 18\end{bmatrix}$$

$$\mathbf{BA}=\begin{bmatrix}-1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

$$=\begin{bmatrix}(-1)1+2\cdot 3 & (-1)2+2\cdot 4\\ 1\cdot 1+3\cdot 3 & 1\cdot 2+3\cdot 4\end{bmatrix}$$

$$=\begin{bmatrix}5 & 6\\ 10 & 14\end{bmatrix}$$

$$\mathbf{CD}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 1\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9 & 0+2+6 & 2+0+3 & 1+0+0\\ 4+10+18 & 0+5+12 & 8+0+6 & 4+0+0\end{bmatrix}$$

$$=\begin{bmatrix}14 & 8 & 5 & 1\\ 32 & 17 & 14 & 4\end{bmatrix}$$

D 和 **C** 矩阵不可相乘，因为 **D** 矩阵的列数与 **C** 矩阵的行数不相等，被称为不可乘矩阵. 由例 4 可以知道，两个矩阵可以相乘的先决条件是第一个矩阵的列与第二个矩阵的行相等，否则，两个矩阵不能相乘；从例 5.5 还可知，$\mathbf{AB}\neq\mathbf{BA}$，也就是说，矩阵相乘的次序是不能互换的，在运算中必须注意.

D and **C** are not conformable for multiplication, because the number

of colomns of **D** matrix is not equal to the number of rows of **C** matrix. From example 5.5, we know that the precondition that two matrices are conformable for multiplication is that the first of the two matrices is multiplied to have the same number of columns as the number of rows in the second. If not, they are not conformable for multiplication. From example 4, we also know that $\mathbf{AB} \neq \mathbf{BA}$, that is, we must therefore be careful of the order of multiplication with matrices, which cannot be changed.

(五) 逆矩阵(Inverse of Matrix)

1. 单位矩阵(Identity Matrices)

如果 n 阶方阵 $\mathbf{I}_{n\times n} = [a_{ij}]$，其中

$$\begin{cases} a_{ij} = 1 & (i = j) \\ a_{ij} = 0 & (i \neq j) \end{cases}$$

则称 **I** 矩阵为 n 阶单位阵.单位阵与同阶任意方阵 **A** 相乘,有如下关系成立:

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

If there is an n square matrix $\mathbf{I}_{n\times n} = [a_{ij}]$, where

$$\begin{cases} a_{ij} = 1 & (i = j) \\ a_{ij} = 0 & (i \neq j) \end{cases}$$

then matrix **I** is called an n identity matrix. When an identity matrix is multiplied by any square matrix **A** with the same order, they have the following relationship:

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

2. 逆矩阵(Inverse of Matrix)

对于任一方阵 **A**,如果有 $\mathbf{A}^{-1}$ 存在,使得

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

则 $\mathbf{A}^{-1}$ 被称为 **A** 矩阵的逆矩阵.

For any square matrix **A**, if there is a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

then $\mathbf{A}^{-1}$ is called the inverse of an **A** matrix.

如果有两个矩阵 **A** 和 **B**,它们满足 $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$,则 **A**,**B** 矩阵称为逆矩阵.

It may be verified that two given matrices **A** and **B** are inverses by showing that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

例 5.6(Example 5.6)

求证矩阵 $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ 是矩阵 $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ 的逆矩阵.

Verify that the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ is $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

解(Solution):

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$\therefore \quad \mathbf{B} = \mathbf{A}^{-1}$

矩阵 **B** 是矩阵 **A** 的逆矩阵.

Matrix **B** is the inverse of matrix **A**.

例 5.7(Example 5.7)

证明 **A** 和 **B** 互为逆矩阵,已知

Show that **A** and **B** are inverses, where

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 2\\ -1 & 1 & 2\\ 1 & -2 & -5\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}1 & -1 & 0\\ 3 & 2 & 2\\ -1 & -1 & -1\end{bmatrix}$$

解(Solution):

$$\mathbf{AB}=\begin{bmatrix}0 & 1 & 2\\ -1 & 1 & 2\\ 1 & -2 & -5\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 3 & 2 & 2\\ -1 & -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0+3-2 & 0+2-2 & 0+2-2\\ -1+3-2 & 1+2-2 & 0+2-2\\ 1-6+5 & -1-4+5 & 0-4+5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathbf{I}$$

$$\mathbf{BA}=\begin{bmatrix}1 & -1 & 0\\ 3 & 2 & 2\\ -1 & -1 & -1\end{bmatrix}\begin{bmatrix}0 & 1 & 2\\ -1 & 1 & 2\\ 1 & -2 & -5\end{bmatrix}$$

$$=\begin{bmatrix}0+1+0 & 1-1+0 & 2-2+0\\ 0-2+2 & 3+2-4 & 6+4-10\\ 0+1-1 & -1-1+2 & -2-2+5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathbf{I}$$

$\because \quad \mathbf{AB}=\mathbf{I}=\mathbf{BA}$

$\therefore \quad \mathbf{B}=\mathbf{A}^{-1}, \quad \mathbf{A}=\mathbf{B}^{-1}$

矩阵 **A** 和矩阵 **B** 互为逆矩阵.

Matrix **A** and matrix **B** are the inverse of each other.

三、矩阵的应用(Application of Matrices)

常用矩阵来求解线性代数方程.

Matrices are often used to solve for linear algebraic equations.

(一) 用等价矩阵求解(Equivalent Matrices)

如果矩阵 $\mathbf{A}_{m\times n}$ 通过改变有限的初等变换能变换成 $\mathbf{B}_{m\times n}$,则称 **A** 与 **B** 为等价矩阵,记为 **A** ~ **B** 或 **B** ~ **A**.其变换过程为:

① 交换矩阵中任意两行(或列);

② 对矩阵中的任一行(或列)乘以一个非零实数;

③ 再将该行(或列)的乘数加到另一行(或列).

If $\mathbf{A}_{m\times n}$ can be changed into $\mathbf{B}_{m\times n}$ by a finite number of elementary operation, we call **A** and **B** equivalent matrices, written **A** ~ **B** or **B** ~ **A**. The operations are as follows:

① Interchange any two rows or columns of a matrix;

② Multiply any row or column of a matrix by a nonzero real number;

③ Then add a multiple of the row or column to another.

利用等价矩阵的性质,可以求解多元线性方程,如下列三元线性代数方程组:

Use the property of equivalent matrices to consider the following system of linear equations in three unknowns:

$$\begin{cases} x + 2y + z = 3 \\ 2x - y - 2z = 3 \\ x + y + 2z = 6 \end{cases}$$

将变量位于方程组左边,常数位于右边.该方程组可以用增广

矩阵来表示，

The variables are lined up on the left side of the system and the constants are on the right side. This system may be represented as the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 2 & -1 & -2 & 3 \\ 1 & 1 & 2 & 6 \end{array}\right]$$

其中系数的矩阵为

where the matrix of coefficients is

$$\left[\begin{array}{rrr} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

解这个方程组，发现求解过程中的每一步都对等于相应矩阵上的初等行运算.用 R_1，R_2，和 R_3 代表增广矩阵的行.右边表示行运算的每一步.

Solving this system, it is seen that each step in the solution is equivalent to an elementary row operation on the corresponding matrix. Let the rows of the augmented matrix be represented by R_1, R_2 and R_3 . The operations on the rows are indicated at the right for each step.

$$\begin{cases} x + 2y + z = 3 \\ 2x - y - 2z = 3 \\ x + y + 2z = 6 \end{cases} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 2 & -1 & -2 & 3 \\ 1 & 1 & 2 & 6 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

方程出现的顺序是任意的，因此它们可以按任意顺序写.所以该方程组等价于

The order in which the equations occur is arbitrary, so they may be written in any order. Thus, the system is equivalent to

$$\begin{cases} x + 2y + z = 3 \\ x + y + 2z = 6 \\ 2x - y - 2z = 3 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 6 \\ 2 & -1 & -2 & 3 \end{array}\right] \begin{matrix} R_1 \\ R_3 \\ R_2 \end{matrix}$$

将第一个方程乘以 -1，再加到第二个方程，得到另一个等价方程组：

The first equation may be multiplied by -1 and added to the second to obtain another equivalent system:

$$\begin{cases} x + 2y + z = 3 \\ -y + z = 3 \\ 2x - y - 2z = 3 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 3 \\ 2 & -1 & -2 & 3 \end{array}\right] \begin{matrix} R_1 \\ (-1)R_1 + R_2 \\ R_3 \end{matrix}$$

再用第一个方程乘以 (-2) 再加到第三个方程，消去 x：

Continuing, x may be eliminatad from the third equation by adding -2 times the first equation to the third equation:

$$\begin{cases} x + 2y + z = 3 \\ -y + z = 3 \\ -5y - 4z = -3 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 3 \\ 0 & -5 & -4 & -3 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ (-2)R_1 + R_3 \end{matrix}$$

用第二个方程乘以 (-5) 再加到第三个方程，消去 y 元素：

The y term may be eliminated from the third equation now by adding -5 times the second to the third:

$$\begin{cases} x + 2y + z = 3 \\ -y + z = 3 \\ -9z = -18 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & -9 & -18 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ (-5)R_2 + R_3 \end{matrix}$$

任何一个方程均可乘以一个非零常数，因此第二个方程乘以 -1，第三个方程乘以 $-\frac{1}{9}$，得：

Any equation may be multiplied by a nonzero constant, so multiply the second equation by -1 and the third equation by $-\frac{1}{9}$ to obtain the following:

$$\begin{cases} x + 2y + z = 3 \\ y - z = -3 \\ z = 2 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ (-1)R_2 \\ -\frac{1}{9}R_3 \end{matrix}$$

由此得 $z = 2$,可将该值代入第二个方程求出 y,再将 z 和 y 的解代入第一个方程求出 x.依次进行,可以消除各未知项,以得到:

This gives the solution $z = 2$, which could be substituted into the second equation to find y, and the solutions of z and y could be substituted into the first to find x. Alternately, you could continue eliminating terms until you obtain

$$\begin{cases} x + 2y = 1 \\ y = -1 \\ z = 2 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 + (-1)R_3 \\ R_2 + R_3 \\ R_3 \end{matrix}$$

最终,

Finally,

$$\begin{cases} x = 3 \\ y = -1 \\ z = 2 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 + (-1)R_2 \\ R_2 \\ R_3 \end{matrix}$$

注意方程组和相应的矩阵

Notice that the system and the corresponding matrix

$$\begin{cases} x + 2y + z = 3 \\ 2x - y - 2z = 3 \\ x + y + 2z = 6 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & -2 & 3 \\ 1 & 1 & 2 & 6 \end{bmatrix}$$

被转换成

were transformed to

$$\begin{cases} x = 3 \\ y = -1 \\ z = 2 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

用初等行运算求解线性代数方程，首先将方程的未知数系数及常数项写成一个增广矩阵形式，然后将这个增广矩阵进行初等运算，每一个用 R_i 描述.

Using elementary row operation for the solution of a system of linear algebraic equations, first rewrite the coefficients of the unknown and constant as an augmented matrix, and then perform the elementary row operations. The operations are chronicled by the subscripted R_i.

例 5.8(Example 5.8) 求(Find) $\begin{cases} 2x + y = 1 \\ 6x + 4y = -2 \end{cases}$

解(Solution):

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 6 & 4 & -2 \end{array}\right] \begin{matrix} R_1 \\ R_2 \end{matrix} \sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & -5 \end{array}\right] \begin{matrix} R_1 \\ R_2 - 3R_1 \end{matrix}$$

$$\sim \left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & -5 \end{array}\right] \begin{matrix} R_1 - R_2 \\ R_2 \end{matrix} \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \end{array}\right] \begin{matrix} \frac{1}{2}R_1 \\ R_2 \end{matrix}$$

$\therefore \quad x = 3, \ y = -5$

例 5.9(Example 5.9) 求(Find) $\begin{cases} x - y = 3 \\ 2x - y + z = 8 \\ y + 2z = 5 \end{cases}$

解(Solution):

$$\left[\begin{array}{ccc|c}1 & -1 & 0 & 3\\ 2 & -1 & 1 & 8\\ 0 & 1 & 2 & 5\end{array}\right]\begin{array}{l}R_1\\ R_2\\ R_3\end{array} \sim \left[\begin{array}{ccc|c}1 & -1 & 0 & 3\\ 0 & 1 & 1 & 2\\ 0 & 1 & 2 & 5\end{array}\right]\begin{array}{l}R_1\\ R_2-2R_1\\ R_3\end{array}$$

$$\sim \left[\begin{array}{ccc|c}1 & 0 & 1 & 5\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 3\end{array}\right]\begin{array}{l}R_1+R_2\\ R_2\\ R_3-R_2\end{array} \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 3\end{array}\right]\begin{array}{l}R_1-R_3\\ R_2-R_3\\ R_3\end{array}$$

$$\therefore \quad x=2,\ y=-1,z=3$$

(二) 用求逆矩阵的方法求线性方程组

(Matrix Solution of Systems of Linear Equations by Inverses)

用矩阵形式,可将线性方程组写成另一种形式.

Using matrix notation, a system of equations can be written in yet another form.

例(For example),假定(suppose)

$$\begin{cases}x+2y+3z=1\\ 2x+5y+4z=2\\ x-3y+2z=3\end{cases}$$

用 **A** 代表方程组中系数的矩阵,**X** 表示未知数矩阵:

Let **A** be the matrix of the coefficients in the system, and let **X** be a matrix of the variables in the system:

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 4\\ 1 & -3 & 2\end{bmatrix},\qquad \mathbf{X}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

注意乘积为

Notice that the product

$$\mathbf{AX}=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 4\\ 1 & -3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{cases}x+2y+3z\\ 2x+5y+4z\\ x-3y+2z\end{cases}$$

是一个有着与方程组左边完全相同的元素的矩阵.这时,如果 **B** 为

方程组右边的常量列矩阵，

is a matrix that has entries exactly like the left side of the system. Now, if **B** is the column matrix of the constants on the right side of the system,

$$\mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

则方程组可以写为矩阵方程 $\mathbf{AX} = \mathbf{B}$. 即：

then the system may be written as the matrix equation $\mathbf{AX} = \mathbf{B}$. That is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

如果矩阵 **A** 有逆矩阵，两边同乘以 $\mathbf{A}^{-1}$ 可求出方程组 $\mathbf{AX} = \mathbf{B}$ 的解.

If matrix **A** has an inverse, the solution of the system $\mathbf{AX} = \mathbf{B}$ may be found by multiplying both sides of the matrix equation by $\mathbf{A}^{-1}$.

$$\mathbf{AX} = \mathbf{B}$$
$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

用初等行运算法将增广矩阵 $[\mathbf{A}|\mathbf{B}]$ 转换成 $[\mathbf{I}|\mathbf{A}^{-1}\mathbf{B}]$，因为 $\mathbf{A}^{-1}\mathbf{B}$ 已经证明是矩阵方程 $\mathbf{AX} = \mathbf{B}$ 的解. 即，

The augmented matrix $[\mathbf{A} \mid \mathbf{B}]$ may be transformed to $[\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{B}]$ by elementary row operation since $\mathbf{A}^{-1}\mathbf{B}$ has proved to be the solutions of the matrix equation $\mathbf{AX} = \mathbf{B}$. That is,

$$[\mathbf{A} \mid \mathbf{B}] \sim [\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{B}]$$

这些相同的行运算法可将 **I** 转换成 $\mathbf{A}^{-1}$：

The same elementary row operations may transform **I** into $\mathbf{A}^{-1}$:

$$[\mathbf{A}|\mathbf{I}] \sim [\mathbf{A}^{-1}\mathbf{A}|\mathbf{A}^{-1}\mathbf{I}] \sim [\mathbf{I}|\mathbf{A}^{-1}]$$

这意味着如果一个矩阵存在逆矩阵，它就可以用初等行运算法通过增加单位矩阵来计算．如果一个矩阵有逆矩阵，这个矩阵称为非奇异矩阵．

This means the inverse of a matrix, if it exists, may be computed by elementary row operations on the matrix augmented by the identity matrix. If a matrix has an inverse, it is called a nonsingular matrix.

例 5.10(Example 5.10)

求矩阵 $\mathbf{A}=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ 的逆矩阵．

Find the inverse of $\mathbf{A}=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$.

解： 先写成增广矩阵$[\mathbf{A}|\mathbf{I}]$：

First write it as an augmented matrix$[\mathbf{A}|\mathbf{I}]$:

$$\begin{bmatrix}2 & 1 & 1 & 0\\ 3 & 2 & 0 & 1\end{bmatrix}\begin{matrix}R_1\\ R_2\end{matrix} \sim \begin{bmatrix}2 & 1 & 1 & 0\\ 1 & 1 & -1 & 1\end{bmatrix}\begin{matrix}R_1\\ R_2-R_1\end{matrix}$$

$$\sim \begin{bmatrix}1 & 0 & 2 & -1\\ 1 & 1 & -1 & 1\end{bmatrix}\begin{matrix}R_1-R_2\\ R_2\end{matrix}$$

$$\sim \begin{bmatrix}1 & 0 & 2 & -1\\ 0 & 1 & -3 & 2\end{bmatrix}\begin{matrix}R_1-R_2\\ R_2-R_1\end{matrix}$$

$[\mathbf{A}|\mathbf{I}]$被转换成$[\mathbf{I}|\mathbf{A}^{-1}]$，其中

$[\mathbf{A}|\mathbf{I}]$ is transformed into $[\mathbf{I}|\mathbf{A}^{-1}]$ where

$$\mathbf{A}^{-1}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$$

例 5.11(Example 5.11)

求矩阵 $\mathbf{B}$ 的逆矩阵．

Find the inverse of matrix $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

解(Solution):

写成增广矩阵$[\mathbf{B}|\mathbf{I}]$，然后用初等行运算法求$[\mathbf{I}|\mathbf{B}^{-1}]$.

Write it as an augmented matrix $[\mathbf{B}|\mathbf{I}]$; then use elementary row operations to obtain $[\mathbf{I}|\mathbf{B}^{-1}]$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_2 - 2R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -1 \\ 0 & 1 & 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] R_1 + R_2$$

$$B^{-1} = \begin{bmatrix} -3 & 2 & -1 \\ -4 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

例 5.12(Example 5.12)

解方程组

Solve the system of equations

$$\begin{cases} 2x + 6 = 1 \\ 3x + 2y = -1 \end{cases}$$

解(Solution):

方程组可用 $\mathbf{AX} = \mathbf{B}$ 的形式表示.

The system may be represented in the form of $\mathbf{AX} = \mathbf{B}$.

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\mathbf{A}$ 的逆矩阵可从例 5.12 中知道,因此 $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

The inverse of $\mathbf{A}$ may be known from example 5.12, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

∴ 方程组的解为$(-3,5)$.

Therefore, the solution of the system is $(3, -5)$.

例 5.13(Example 5.13)

解方程组

Solve the system of equations

$$\begin{cases} x - y = 3 \\ 2x - y + z = 8 \\ y + 2z = 5 \end{cases}$$

解(Solution):

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$$

已知系数矩阵的逆矩阵,未知数列阵可写为

Given the inverse of a matrix of the coefficients, the column matrix of unknown may be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 \\ -4 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

∴ 方程组的解$(x, y, z) = (2, -1, 3)$.

Therefore, the solution of the system $(x, y, z) = (2, -1, 3)$.

这种逆矩阵法以及其它矩阵法的真正优势在于计算机的使用.

The real advantage of this inverse method, as well as other matrix methods, is the use of the computer.

§5.2 行列式的表达(Determinants)

一、定义(Definitions)

行和列都相等的矩阵称为方阵,由 n 阶方阵形式的 n^2 个数可以确定一个实数,每一个有实数项的 $A_{n\times n}$ 都具有一个唯一的实数,这个实数被称为行列式,记为 $\delta(A)$ 或 $|A|$. $\delta(A_{n\times n})$ 也称为 n 阶行列式.

A matrix with the same number of rows and columns is called a square matrix. n^2 numbers in a nth-order matrix may determine a real number. With every square matrix $A_{n\times n}$ with real number entries, there is associated a unique real number called its determinant, denoted by $\delta(A)$ or $|A|$. Also, $\delta(A_{n\times n})$ is called a determinant of order n.

记号

Symbol

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = |a_{ik}| = \Delta$$

称为 n 阶行列式.

is called a determinant of order n.

行列式的横排称为行,竖排称为列,A_{ij} 称为行列式的元素.A_{ij} 的第一个下角标代表行,第二个下角标代表列,表明该元素在行列式中所处的位置.

The horizontal entries of a determinant is called rows and the vertical ones columns. A_{ij} is called the members of the determinant. The first subscript of A_{ij} represents the row and the second the column, which indicate the position of the members in the determinant.

(一) 二阶行列式(Determinants of Order 2)

二阶行列式为

A determinant of order 2 is

$$\delta(A_{2\times 2}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

二阶行列式等于交差相乘的差:

A determinant of order 2 is the difference of cross products:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

例 5.14 如果 $\mathbf{A} = \begin{bmatrix} 7 & -5 \\ 2 & 3 \end{bmatrix}$,求 $\delta(A)$.

解： $\delta(A) = \begin{bmatrix} 7 & -5 \\ 2 & 3 \end{bmatrix} = (7)(3) - (2)(-5)$

$= 21 - (-10)$

$= 31$

Example 5.14 If $A = \begin{bmatrix} 7 & -5 \\ 2 & 3 \end{bmatrix}$, find $\delta(A)$.

Solution $\delta(A) = \begin{bmatrix} 7 & -5 \\ 2 & 3 \end{bmatrix} = (7)(3) - (2)(-5)$

$= 21 - (-10) = 31$

因为行列式与矩阵很类同，必须注意区分.矩阵是一种数的排列，但行列式则是由排列所决定的实数值.每个方阵都有一个相关的行列式，但行列式的次数超过二阶时，求解它的值就不太容易了.

Since determinants resemble matrices, be careful to distinguish between them. Matrices are arrays of numbers, and determinants are real numbers defined by an array. Every square matrix has an associated determinant, but determinants of order greater than 2 are not evaluated as easily as determinants of order 2.

(二) 三阶行列式(Determinants of Order 3)

一个三阶行列式可定义为

A determinant of order 3 is defined as

$$\delta(A_{3\times 3}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

该式显得相当长，但可通过提取公因式化简：

This looks pretty involved, but it may be simplified by abstracting common factors:

$$\begin{aligned}\delta(A_{3\times3}) &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32} - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\end{aligned}$$

现在 3×3 阶行列式用 2×2 阶行列式来表达.

The 3 × 3 determinant is now expressed in terms of 2 × 2 determinants.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

例 5.15(Example 5.15)

$$\begin{aligned}\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} &= 1\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} \\ &\quad + (-1)\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \\ &= 1(-2) - 2(2) + (-1)(1) \\ &= -2 - 4 - 1 = -7\end{aligned}$$

$$\delta(A_{3\times 3}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

重新提取公因式.

Simplify it by abstracting the common factor again.

$$\delta(A_{3\times 3}) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

这就给出了关于第一列而不是第一行的行列式的扩展式.

This gives the expansion of the determinant about the first column instead of the first row.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

根据 2×2 行列式表达 3×3 行列式的方式被称为子式的展开方式.

Expressing the 3×3 determinant in terms of 2×2 determinants is called a form of expansion by minors.

(三) 子式(Minors)

a_{ij} 的子式是通过消去 $\delta(A_{n\times n})$ 中的第 i 行和第 j 列而获得的

M_{ij} 行列式.

The minor of a_{ij} is the determinant M_{ij} obtained by deleting the ith row and the jth column of $\delta(A_{n\times n})$.

如在行列式

Such as in the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ 中

$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ 是元素 a_{11} 的子式

is the minor of a_{11}

$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ 是元素 a_{12} 的子式.

is the minor of a_{12}

行列式的每个元素只有一个子式.

Each element of a determinant has only one minor.

例 5.16(Example 5.16) $|A| = \begin{vmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ -1 & -3 & -2 \end{vmatrix}$

找出 a_{11}, a_{32}和 a_{23} 的子式.

Find the minors of a_{11}, a_{32} and a_{23}.

解(Solution):

$a_{11} = 1$; $\begin{vmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ -1 & -3 & -2 \end{vmatrix}$ $M_{11} = \begin{vmatrix} 3 & 4 \\ -3 & -2 \end{vmatrix} = 6$

$a_{32} = -3$; $\begin{vmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ -1 & -3 & -2 \end{vmatrix}$ $M_{32} = \begin{vmatrix} 1 & -4 \\ 0 & 4 \end{vmatrix} = 4$

$$a_{23} = 4; \quad \begin{vmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ -1 & -3 & -2 \end{vmatrix} \quad M_{11} = \begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} = -1$$

(四) 余子式(Cofactors)

求解行列式的值时,用子式求解有时可简化计算,但要考虑子式所带的正负号.带有正负号的子式称为元素的余子式.

In obtaining the value of a determinant, it is possible to simplify the operation by using minors, but we must consider the positive and negative signs affixed to the minors. The minor with the proper sign affixed is called the cofactor of the corresponding element.

行列式 A_{ij} 是 a_{ij} 的余子式,且

$$A_{ij} = (-1)^{i+j}M_{ij}$$

当 $i + j$ 为偶数时,a_{ij} 的余子式是 M_{ij};当 $i + j$ 为奇数时,a_{ij} 的余子式是 $-M_{ij}$.

The determinant A_{ij} is the cofactor of a_{ij} and

$$A_{ij} = (-1)^{i+j}M_{ij}$$

The cofactor of a_{ij} is M_{ij} if $i + j$ is even and $-M_{ij}$ if a_{ij} is odd.

例 5.17(Example 5.17) $\quad |A| = \begin{vmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ -1 & -3 & -2 \end{vmatrix}$

求 A_{12}, A_{31} 和 A_{23}.

Find A_{12}, A_{31} and A_{23}.

解(Solution):

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 4 \\ -1 & -2 \end{vmatrix} = (-1)(4) = -4$$

2 的余子式是 -4.

The cofactor of 2 is -4.

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -4 \\ 3 & 4 \end{vmatrix} = (+1)(20) = 20$$

-1 的余子式是 20.

The cofactor of -1 is 20.

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} = (-1)(-1) = 1$$

4 的余子式是 1.

The cofactor of 4 is 1.

高阶行列式可以按某行或列元素展开为低阶行列式再进行求解.行列式的展开式等于按某行(或列)各元素与其余子式乘积之和.

A higher order determinant may be expanded to a lower order determinant in terms of the elements in a row or column in evaluating the determinant. The expansion of a determinant equals the sum of the products between each element and its cofactor in that row or column.

例 5.18(Example 5.18)

对下列行列式按第三列展开.

Expand the following determinant about the third column.

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix}$$

解(Solution):

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} = (-1)^{1+3}(-1)\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

$$+(-1)^{2+3}(2)\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}+(-1)^{3+3}(-2)\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}$$
$$=(-1)\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}-2\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}+(-2)\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}$$
$$=(-1)(1)-2(2)+(-2)(1)$$
$$=-1-4-2=-7$$

例 5.19(Example 5.19)

对下列行列式按第一行展开.

Expand the following determinant about the first row.

$$\begin{vmatrix}1 & 3 & -2\\ 1 & -4 & 5\\ 2 & -2 & 1\end{vmatrix}$$

解(Solution):

$$\begin{vmatrix}1 & 3 & -2\\ 1 & -4 & 5\\ 2 & -2 & 1\end{vmatrix}=1\begin{vmatrix}-4 & 5\\ -2 & 1\end{vmatrix}-3\begin{vmatrix}1 & 5\\ 2 & 1\end{vmatrix}+(-2)\begin{vmatrix}1 & -4\\ 2 & -2\end{vmatrix}$$
$$=1(6)-3(-9)+(-2)(6)$$
$$=6+27-12=21$$

二、行列式的性质(Properties of Determinants)

运用行列式的性质,可以简化求解行列式的过程,特别是对于那些高阶行列式.

Because of the properties of determinants, a process of simplification can be developed to help evaluate determinants, especially those of large order.

性质 1(Property 1)

如果行列式中有两行或两列完全相同,则行列式的值等于0.

If two rows or two columns of a determinant are identical, the value of the determinant is zero.

例5.20(Example 5.20)　按第一行展开行列式.

Expand about the first column.

$$\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ -3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0 + (-3)(0) = 0$$

性质2(Property 2)

如果行列式中有一行或列的所有元素都为0,则行列式的值等于零.

If one row or column has all zero entries, the value of the determinant is zero.

例5.21(Example 5.21)　按第二列展开.

Expand about the second column.

$$\begin{vmatrix} 1 & 0 & -3 \\ 2 & 0 & 2 \\ 3 & 0 & 1 \end{vmatrix} = -0\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix}$$

$$= 0 + 0 - 0 = 0$$

性质3(Property 3)

如果行列式某一行或列的每一个元素乘以一常数,则行列式的值乘以这个常数.

If each entry in one row or one column is multiplied by a constant, the value of the determinant is multiplied by that constant.

例 5.22(Example 5.22)　按第一行展开下列行列式.

Expand about the first row of the following determinant.

$$\begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 4\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= 1(2) - 0 + 4(-7)$$

$$= 2 - 0 - 28 = -26$$

将以上行列式第一列各元素乘以 2,然后按第一行展开.

Multiply the first column of the above determinant by 2 and expand about the first row.

$$\begin{vmatrix} 2 & 0 & 4 \\ -2 & 2 & 0 \\ 6 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 4\begin{vmatrix} -2 & 2 \\ 6 & 1 \end{vmatrix}$$

$$= 2(2) - 0 + 4(-14)$$

$$= 4 - 0 - 56 = -52$$

因此,第一列各元素乘以 2,行列式的值也将扩大一倍.

Thus, multiplying the first column by 2 doubles the value of the determinant.

性质 4(Property 4)

如果某一行(列)的所有元素的倍数分别加上相应的另一行(列)的所有元素,则行列式的值不变.

If a multiple of the entries of one row/column is added to corresponding entries of another row/column, respectively, the value of the determinant remains the same.

例 5.23(Example 5.23)　根据上例,

According to the above example,

$$(1)\quad \begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4+(-4) \\ -1 & 2 & 0+(4) \\ 3 & 1 & 1+(-12) \end{vmatrix}$$
$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix} \qquad \begin{matrix} C_1 & C_2 & C_3 - 4C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 4 \\ 3 & 1 & -11 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 4 \\ 1 & -11 \end{vmatrix} - 0 + 0$$

$$= -22 - 4 = -26$$

$$(2)\quad \begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} = \begin{vmatrix} 1-1 & 0+2 & 4+0 \\ -1 & 2 & 0 \\ 3-3 & 1+6 & 1+0 \end{vmatrix} \begin{matrix} R_1+R_2 \\ R_2 \\ R_3+3R_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 2 & 4 \\ -1 & 2 & 0 \\ 0 & 7 & 1 \end{vmatrix} = 0 - (-1)\begin{vmatrix} 2 & 4 \\ 7 & 1 \end{vmatrix} + 0$$

$$= 0 + (-26) + 0 = -26$$

运用以上性质可以简化行列式的计算.

Using the properties above may simplify the operation of a determinant.

例 5.24(Example 5.24)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-3(3) & 2 & 3 \\ 4-3(-2) & -1 & -2 \\ 3-3(1) & 0 & 1 \end{vmatrix}$$
$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix} \qquad \begin{matrix} C_1-3C_3 & C_2 & C_3 \end{matrix}$$

$$= \begin{vmatrix} -8 & 2 & 3 \\ 10 & -1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 0 - 0 + 1 \begin{vmatrix} -8 & 2 \\ 10 & -1 \end{vmatrix}$$

$$= 1(8 - 20) = -12$$

例 5.25(Example 5.25)

$$\begin{vmatrix} 3 & -2 & -1 & 2 \\ 4 & 1 & 2 & -3 \\ -9 & -5 & 7 & -8 \\ 1 & 5 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 10 & -3 & 2 & 1 \\ 12 & -19 & 7 & 6 \\ 10 & -1 & 3 & 4 \end{vmatrix}$$

$C_1 \quad C_2 \quad C_3 \quad C_4 \qquad C_1 - 3C_3 \; C_2 - 2C_3 \; C_3 \qquad C_4 + 2C_3$

$$= (-1)\begin{vmatrix} 10 & -3 & 1 \\ 12 & -19 & 6 \\ 10 & -1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 10 & -3 & 1 \\ -48 & -1 & 0 \\ -30 & 11 & 0 \end{vmatrix} \begin{matrix} R_1 \\ R_2 - 6R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$= (-1)(1)\begin{vmatrix} 48 & -1 \\ -30 & 11 \end{vmatrix} = (-1)(1)(-528 - 30) = 558$$

例 5.26(Example 5.26)

解行列式中的 x (Solve for x of the determinant):

$$\begin{vmatrix} 1 & -3 & 1 \\ -1 & x & -2 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

按第二行展开.

Expand about the second row.

$$-(-1)\begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} + x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = 0$$

$$(-1)(-8) + x(1) - (-2)(5) = 0$$

$$8 + x + 10 = 0$$
$$x = -2$$

三、用行列式求解线性代数方程组
(Determinant Solution of Systems)

考虑用增广矩阵表示的线性代数方程：

Recall the system of linear equations represented by an augmented matrix:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \quad \begin{cases} a_{11}x + a_{12}y = b_1 & ① \\ a_{21}x + a_{22}y = b_2 & ② \end{cases}$$

方程 ① 两边同乘 a_{22}.

Multiply both sides of equation ① by a_{22}.

$$a_{11}a_{22}x + a_{12}a_{22}y = a_{22}b_1$$

方程 ② 两边同乘 $-a_{12}$.

Multiply both sides of equation ② by $-a_{12}$.

$$-a_{12}a_{21}x - a_{12}a_{22}y = a_{12}b_2$$

两边相加.

Add the two equations.

$$(a_{11}a_{22} - a_{12}a_{21})x = a_{22}b_1 - a_{12}b_2$$

如果(If) $(a_{11}a_{22} - a_{12}a_{21})x \neq 0$

则(Then)：

$$x = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

同理(Similarly)：
$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

我们把 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 称为方程的系数 x, y 的行列式，记为 D. 将系数行列式中 x 的系数用常数项 b_i 所取代，得到的行列式记为 D_x. 将系数行列式中 y 的系数用常数 b_i 所取代，得到的行列式记为 D_y.

We call $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ the determinant of the coefficients of x and y of the system, denoted by D. The x column of D is replaced by the constant b_i, and the determinant obtained is denoted by D_x. The coefficient of y is replaced by the constant b_i and the resulting determinant is denoted by D_y.

克莱姆法则(Gramer's Rule)

如果 $\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$ 是线性方程组中 x, y 的增广矩阵，则

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}, \qquad D \neq 0$$

If $\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$ is the augmented matrix of a linear system in x and y, then

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}, \qquad D \neq 0$$

例 5.27(Example 5.27) 解线性方程组求 x 和 y：

Solve the system for x and y :

$$\begin{cases} 4x - 8y = 17 \\ 12x + 16y = -9 \end{cases}$$

解(Solution):

先形成增广矩阵:

First, form the augmented matrix:

$$\left[\begin{array}{cc|c} 4 & -8 & 17 \\ 12 & 16 & -9 \end{array}\right]$$

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 17 & -8 \\ -9 & 16 \end{vmatrix}}{\begin{vmatrix} 4 & -8 \\ 12 & 16 \end{vmatrix}} = \frac{200}{160} = \frac{5}{4}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 17 \\ 12 & -9 \end{vmatrix}}{160} = \frac{-240}{160} = -\frac{3}{2}$$

$$\therefore \quad (x, y) = \left(\frac{5}{4}, -\frac{3}{2}\right).$$

克莱姆法则可以被应用于 n 阶线性方程.

Cramer's rule is general and can be applied to a linear system of n equations in n variables.

如果(If)

$$\left[\begin{array}{ccccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & & b_i \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} & & b_n \end{array}\right]$$

是关于 $x_1, x_2, \cdots, x_i, \cdots x_n$ 的线性代数方程组的增广矩阵,则:

is the augmented matrix of a linear system in $x_1, x_2, \cdots, x_i, \cdots x_n$, then

$$x_1 = \frac{D_1}{D}, \; x_2 = \frac{D_2}{D}, \cdots \; x_i = \frac{D_i}{D}, \cdots \; x_n = \frac{D_n}{D} \quad \text{且(and)} \; D \neq 0$$

例 5.28(Example 5.28)　求解线性方程组

Solve for the linear system of equations

$$\begin{cases} x - 2y - 3z = 3 \\ x + y - z = 2 \\ 2x - 3y - 5z = 5 \end{cases}$$

解(Solution):

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 3 \\ 1 & 1 & -1 & 2 \\ 2 & -3 & -5 & 5 \end{array}\right]$$

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 3 & -2 & -3 \\ 2 & 1 & -1 \\ 5 & -3 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & -3 \\ 1 & 1 & -1 \\ 2 & -3 & -5 \end{vmatrix}} = \begin{vmatrix} -3 & -5 & -3 \\ 0 & 0 & -1 \\ -5 & -8 & -5 \end{vmatrix} \Big/ \begin{vmatrix} -2 & -5 & -3 \\ 0 & 0 & -1 \\ -3 & -8 & -5 \end{vmatrix}$$

$$= \frac{-(-1)\begin{vmatrix} -3 & -5 \\ -5 & -8 \end{vmatrix}}{-(-1)\begin{vmatrix} -2 & -5 \\ -3 & -8 \end{vmatrix}} = \frac{24 - 25}{16 - 15} = \frac{-1}{1} = -1$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 1 & 3 & -3 \\ 1 & 2 & -1 \\ 2 & 5 & -5 \end{vmatrix}}{1} = \frac{\begin{vmatrix} 1 & 3 & -3 \\ 0 & -1 & 2 \\ 0 & -1 & 1 \end{vmatrix}}{1}$$

$$= 1\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} = -1 + 2 = 1$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 1 & -2 & 3 \\ 1 & 1 & 2 \\ 2 & -3 & 5 \end{vmatrix}}{1} = \frac{\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & -1 \\ 0 & 1 & -1 \end{vmatrix}}{1}$$

$$= \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\therefore \quad (x, y, z) = (-1, 1, -2)$$

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第六章　三角形的图形表达（Triangle Graphs）

§6.1　线和角(Lines and Angles)

一、线的表达(Lines)

(一) 点(Points)

几何上的点是空间里的一个确定位置.它没有尺寸,也不可见.但它是几何位置的唯一表达. 位于同一直线上的点称为共线点.点常用大写字母来表示. 如 **R**,读作"点 **R**".

A geometric point is an element of geometry which has a position but no extension. It has no size, nor can it be seen. It is a single expression of a geometric position. Points that lie on the same straight line are called collinear points. A capital letter is often used to mark and name a point. For instance, **R** is read "point **R**".

(二) 直线(Straight Lines)

几何直线是共线点的集合. 如铅笔和粉笔都可看作直线.几何线中的直线是由无数个点集合而成的,它没有端点,可沿两个方向无限延伸. 直线的任意部分没有宽度,但有长度.

A geometric straight line is a set of collinear points; for instance, pencils and chalk can be both regarded as straight lines. A geometric line

has an infinite number of points without endpoints and may be extended indefinitely in both directions. Any part of a line has no width or thickness but length.

命名一条直线可用一个小写字母表示,如图 6.1 所示的直线可记作直线 l;一条直线也可以用在这条直线上的任意两点来表示,如图 6.1 所示的直线也可以记作直线 AB,且图 6.1 所示的直线可以读作"直线 AB",也可以读作"直线 BA",它们的次序可以互换.



图 6.1(Figure 6.1)

To name a line, we can express it in a small letter, such as the line in figure 6.1, which can be written as "line l"; we can also use any two labeled points on the straight line to express the line, so the line in figure 6.1 can also be written as "line AB" or "line BA", the order of which can be interchanged.

直线公理(Axioms of Straight Lines):

公理 1(Axiom 1)

过两点有一条且只有一条直线.

There is one and only one straight line through two points.

公理 2(Axiom 2)

每条直线上有无穷多个点.

There are indefinite points on a straight line.

公理 3(Axiom 3)

一条直线上任意三点中,有且只有一点在另两点之间.

Among any three points on a straight line, there is one and only one

point between two other points.

公理 4(Axiom 4)

如果 A 和 B 是直线上的两点,则直线上存在无限多个点在 A 和 B 之间.

If A and B are two points on a straight line, then there are indefinite points between points A and B .

(三) 射线和线段(Rays and Line Segments)

射线和线段都是直线的一部分.

A ray and a line segment are both a part of a straight line.

1. 射线(Rays)

直线上任一点可将直线分为两部分.点和它一旁的部分叫做**射线或半直线**,这个点叫做射线的**端点**.射线有一个端点,并向一个方向无限延伸.

Any point on a straight line can separate the line into two parts. The part extending from the point is called a **ray** or a **half straight line** and the point is called an **endpoint.** A ray has one endpoint and extend to one direction indefinitely.

一条射线可以用端点和射线上的另一个点来表示,如图 6.2 所示的射线,可记作"射线 OA ". 特别应注意,表示端点的字母要写在前面.一条射线也可以用一个小写字母来表示,如图 6.2 中的射线 OA 也可以记作"射线 l ".

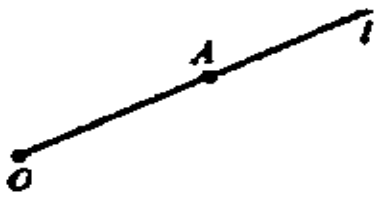


图 6.2(Figure 6.2)

A ray can be expressed by an endpoint and another point on the ray.

For instance, the ray in figure 6.2 is named "ray OA". What should be particularly noticed is that the letter denoting an endpoint is written in front of the ray. A ray can also be expressed by a small letter, such as the ray OA in figure 6.2, which may be written as "ray l", as well.

2. 线段(Line Segments)

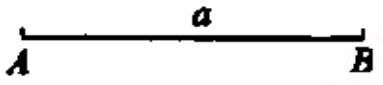


图 6.3(Figure 6.3)

直线上的两个点和它们之间的部分叫做**线段**,这两个点叫做线段的**端点**.线段有两个端点,且不能延伸,长度的值是确定的.线段用两个端点来表示,如图 6.3 所示,两端点为 A,B,则该线段记作线段 AB 或线段 BA,有时也可记作线段 a.

A definite part of a line between two points of a straight line is called a **line segment** and the two points are called the **endpoints** of the segment. A line segment has two endpoints and cannot extend with a definite value of length. If a line is expressed by two endpoints A and B, as is shown in figure 6.3, the segment is written as segment AB or BA and sometimes as segment a, as well.

(四) 线的种类和位置(Kinds and Positions of Lines)

1. 线的种类(Kinds of Lines)



(1) 直线(Straight Line) (2) 曲线(Curved Line) (3) 折线(Broken Line)

图 6.4(Figure 6.4)

除直线外,还有曲线和折线,如图 6.4 所示.

Besides the straight lines, there are also curved lines and broken lines, as is shown in figure 6.4.

由不是共线点的集合组成的线为曲线. 由 n 根不同线段端点相互连接组成的线称为折线.

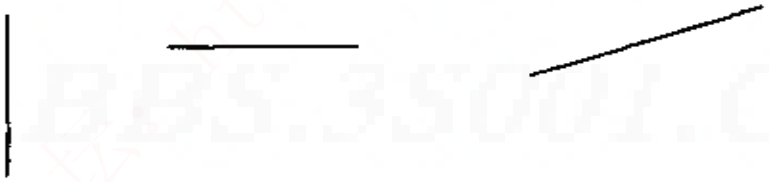
A line composed of sets of uncommon points is called a curved line. A line composed of n different segments connected with each other is called a broken line.

2. 直线的位置(Positions of Lines)

1) 一根直线的位置(Position of One Straight Line)

一根直线在平面中的位置有垂直、水平和倾斜放置三种方式. (见图 6.5)

One straight line in a plane may be in vertical, horizontal and slant or oblique positions. (see figure 6.5)



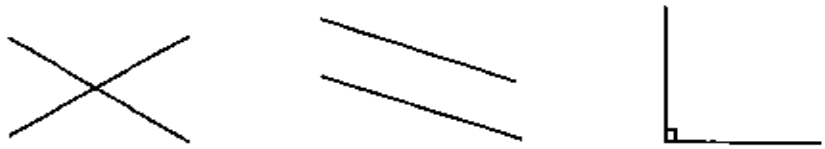
(1) 垂直放置(Vertical) (2) 水平放置(Horizontal) (3) 倾斜放置(Slant)

图 6.5(Figure 6.5)

2) 两根直线的相互位置(Positions of Two Straight Lines)

两根直线在平面中的相互位置有三种:相交、平行和垂直. 如图 6.6 所示.

There are three positions of each other for two lines in a plane: intersecting, parallel and perpendicular, as is shown in figure 6.6.



(1) 相交(Intersecting)　(2) 平行(Parallel)　(3) 垂直(Perpendicular)

图 6.6(Figure 6.6)

当两直线在平面中有一个交点,称为两直线相交.

If two lines have one intersecting point on a plane, the two lines are said to be intersected.

两直线在平面中没有交点,称为两直线平行,用符号 // 表示.

If two lines have no intersecting point in a plane, the two lines are said to be parallel to each other, expressed in a symbol as //.

当两直线在平面中有一个共同点且相互间的角度为直角,则两直线垂直,用符号⊥表示.

It two lines have one common point and the angle between them is perpendicular, the two lines are said to be perpendicular to each other, expressed in a symbol as ⊥.

平面上两点之间可以有无数种类型的线,如图 6.7 所示,但是联结两点的线中,线段最短. 连接两点的线段的长度叫做这**两点间的距离**.



图 6.7(Figure 6.7)

There may be numerous types of lines between two points in a

plane, as is shown in figure 6.7. However, a line segment is the shortest of all the lines between two points. The length of the segment between two points is called the **distance between two points**.

二、角的表达(Angles)

(一) 定义(Definitions)

由公共端点的两条射线组成的图形叫做**角**,这个公共的端点叫做角的**顶点**,这两条射线叫做角**的边**,如图6.8中角的顶点是 O,边是射线 OA、OB.



图 6.8(Figure 6.8)

The figure by two rays drawing from a common endpoint is called an **angle**. This common endpoint is called the **vertex** of the angle and the two rays the **sides** of the angle. As is shown in figure 6.8, the vertex of the angle is O and the sides are OA and OB .

角还可看成是一条射线绕着它的端点从一个位置旋转到另一个位置所形成的图形.如图6.8,可看成射线 OA 绕 O 点从 OA 旋转到 OB 所形成的图形.

An angle may also be regarded as a figure formed by a ray revolving about a fixed point from one position to another. As is shown in figure 6.8, the angle can be treated as a figure formed by ray OA turning about the point O up to OB .

(二) 角的表达方式(Ways of Expressing Angles)

我们通常用符号"∠"来表示"角".

The symbol "∠"is usually used to denote an "angle".

角的表示方法通常有三种(如图 6.9).

1. 用顶点的大写字母来表示角.

2. 用三个大写字母表示,中间的字母必须是角的顶点.

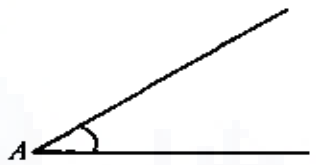
3. 用一个写在角内侧的小写字母或数字来表示.

Angles are usually expressed in three ways (see figure 6.9).

1. Use a capital letter at the vertex.

2. Use three capital letters, of which the middle letter must indicate the vertex of the angle.

3. Use a small letter or figure inside the angle.



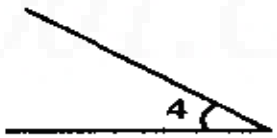
(1) 读作"角 A"
(Read "angle A")



(2) 读作"角 ABC" 或"角 CBA"
(Read "angle ABC" or "angle CBA")



(3) 读作"角 m"
(Read "angle m")



(4) 读作"角 4"
(Read "angle 4")

图 6.9(Figure 6.9)

(三) 角的种类(Kinds of Angles)

如图 6.10 所示,射线 OA 绕点 O 旋转,当终止位置 OC 和起始位置 OA 成一直线时,射线绕 O 点旋转半周,这个角值为 180°,称为

平角.当射线继续旋转,回到起始位置时,所成的角为**周角**,其角为 360°.

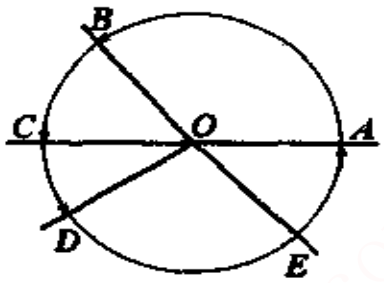
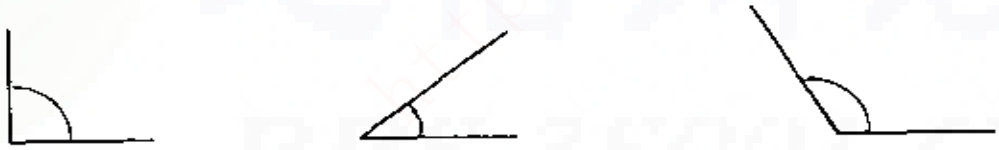


图 6.10(Figure 6.10)

As is shown in figure 6.10, when a ray turns about point O from the position OA up to OC, the finishing position OC and the starting position OA become a straight line in an angle of 180°, which is called a straight angle. When the ray keeps on rotating back to the starting position in a complete circle, the angle formed is called the circumferential angle whose measure is 360°.

90°的角称为**直角**;大于 0°小于 90°的角称为**锐角**;大于 90°小于 180°的角称为**钝角**;大于 180°小于 360°的角称为**优角**,如图 6.11 示.



(1) 直角(Right Angle) (2) 锐角(Acute Angle) (3) 钝角(Obtuse Angle)



(4) 平角(Straight Angle) (5) 优角(Reflex Angle)

6.11(Figure 6.11)

The angle whose measure is 90° is called a right angle; the one whose measure is greater than 0° and less than 90° is an acute angle; the one whose measure is greater than 90° and less than 180° is an obtuse angle; the one whose measure is greater than 180° and less than 360° is a reflex angle, as is shown in figure 6.11.

(四) 角与角之间的关系(Relationship between Angles)

1. 互为补角(Supplementary Angles)

如果两角和等于 180°,则两角**互为补角**.如图 6.12(1),∠1 的补角为∠2,∠2 的补角为∠1.

Two angles are **supplementary** if their sum is 180°. For instance, in figure 6.12(1), the supplementary angle of ∠1 is ∠2 and that of ∠2 is ∠1.

2. 互为余角(Complementary Angles)

如果两个角的和为 90°,则这两个角**互为余角**(如图 6.12(2).例∠3 的余角为∠4,∠4 的余角为∠3.

Two angles are **complementary** if their sum is 90°(see figure 6.12 (2)). For instance, the complementary angle of ∠3 is ∠4 and that of ∠4 is ∠3.



(1) 互为补角(Supplementary)　　(2) 互为余角(Complementary)

图 6.12(Figure 6.12)

三、相交线和平行线(Intersecting and Parallel Lines)

(一) 对顶角和邻补角
(Vertical and Adjacent Supplementary Angles)

如图 6.13 所示,两直线相交得到四个角,这些角有一个公共顶点,其中有些有公共边,有些没有公共边.

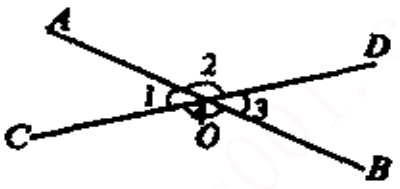


图 6.13(Figure 6.13)

As is shown in figure 6.13, there are four angles when two lines are intersected. These angles have a common vertex; some of the angles have a common side and some haven't.

有一个公共顶点,没有公共边,像这样的两个角叫做**对顶角**,在图 6.13 中,∠2 与∠4; ∠1 与∠3 都是对顶角.

Two angles having a common vertex but no common side are called **vertical angles**, such as, in figure 6.13, ∠2 and ∠4, ∠1 and ∠3 which are all vertical angles.

不仅有一个公共顶点,还有一个公共边,这样的两个角叫做**邻补角**.如∠1 与∠2,∠2 与∠3,∠1 与∠4,∠3 与∠4.(图 6.13)

Two angles having not only a common vertex hat a common side are called **adjacent supplementary angles**, such as ∠1 and ∠2, ∠2 and ∠3, ∠1 and ∠4, ∠3 and ∠4. (see figure 6.13)

两直线相交存在着下列关系:

(Relationships between two straight lines intersected)

1．两直线相交，对顶角相等．

Their vertical angles are identical if two lines are intersected.

2．两直线相交，邻补角互补．

Their adjacent supplementary angles are supplementary if two lines are intersected.

（二）垂线（Perpendicular Lines）

如图 6.14 所示，两条直线相交所成的四个角中，有一个角是直角时，就说这两条直线相互垂直，其中一条直线叫做另一条直线的**垂线**．它们的交点 O 叫做**垂足**．过一点只有一条直线与已知直线垂直．

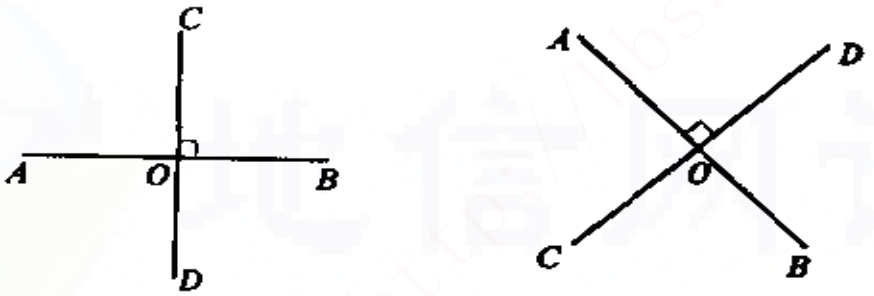


图 6.14（Figure 6.14）

As is shown in figure 6.14, if one of the four angles formed by two lines intersecting each other is a right angle, the two lines may be regarded as perpendicular to each other, one line being another's **perpendicular line.** The intersecting point O is called a perpendicular base, through which only one straight line is perpendicular to a given straight line.

（三）同位角、内错角和同旁内角（Corresponding Angles，Alternate and Same-side Interior Angles）

如图 6.15 所示，直线 AB、CD 与 EF 相交，构成了 8 个角.

As is shown in figure 6.15, lines AB and CD intersect EF, forming eight angles.

位置相同的一对角叫做**同位角**，如 $\angle 1$ 与 $\angle 5$，$\angle 2$ 与 $\angle 6$，$\angle 3$ 与 $\angle 7$，$\angle 4$ 与 $\angle 8$.

A pair of angles in the same position are called **corresponding angles**, such as $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$.

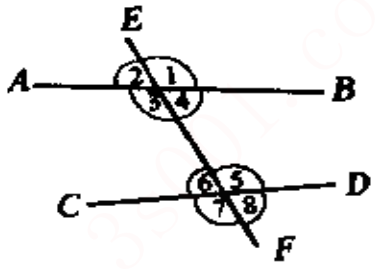


图 6.15(Figure 6.15)

在两直线 AB、CD 之内，并在直线 EF 两侧不相邻的一对内角称为**内错角**，如 $\angle 3$ 和 $\angle 5$，$\angle 4$ 和 $\angle 6$.

A pair of nonneighboring interior angles inside the two lines AB and CD, as well as on both sides of line EF, are called **alternate interior angles**, such as $\angle 3$ and $\angle 5$, $\angle 4$ and $\angle 6$.

在两直线 AB、CD 之内，并在直线 EF 同一旁的不相邻**内角**，称为**同旁内角**，如 $\angle 3$ 与 $\angle 6$，$\angle 4$ 与 $\angle 5$.

A pair of interior angles which are not adjacent inside two lines AB and CD, as well as on the same side of line EF, are called the same-side interior angles, such as $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$.

（四）平行线的性质（Properties of Parallel Lines）

在同一个平面内，不相交的两条直线叫做**平行线**，平行用符号“//”表示，如图 6.16，直线 l_2 与 l_3 是平行线，记作“$l_2 // l_3$”，读

作“l_2 平行于 l_3”. 在同一平面内,两条直线的位置只有相交、平行两种.

Two lines are parallel, denoted as “//”, if there is a plane in which they both lie and they do not meet however far they are produced, such that in figure 6.16, lines l_2 and l_3 are parallel lines, written as “l_2 // l_3”, read “l_2 is parallel to l_3”. In the same plane, two straight lines are either intersecting or parallel.

如图 6.16 所示,两条直线 l_2、l_3被第三条直线 l_1所截,如果 l_2 // l_3,则同位角相等, 如图 6.16,$\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$; 内错角相等,如图 6.16, $\angle 3 = \angle 5$, $\angle 4 = \angle 6$; 同旁内角互补. 如图 6.16,$\angle 3 + \angle 6 = 180^\circ$,$\angle 4 + \angle 5 = 180^\circ$.

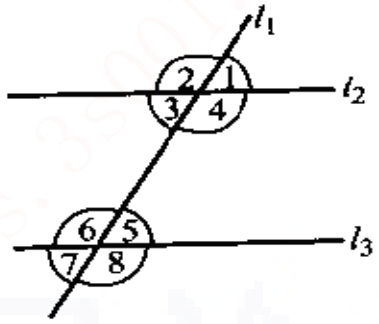


图 6.16(Figure 6.16)

As is shown in figure 6.16, when two lines l_2 and l_3 are cut by a third line l_1, and l_2 // l_3, then the corresponding angles are equal, such as in figure 6.16, $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$; alternate interior angles are equal, such as in figure 6.16, $\angle 3 = \angle 5$, $\angle 4 = \angle 6$; the same-side interior angles are supplementary. For instance, $\angle 3 + \angle 6 = 180^\circ$, $\angle 4 + \angle 5 = 180^\circ$.

§6.2 三角形(Triangles)

一、基本概念(Basic Concepts)

(一) 定义(Definitions)

由不在同一条直线上的三条线段首尾顺次相接组成的图形叫做**三角形**，用符号“△”表示，如图 6.17 所示的三角形记作“△ABC”，读作“三角形 ABC”.

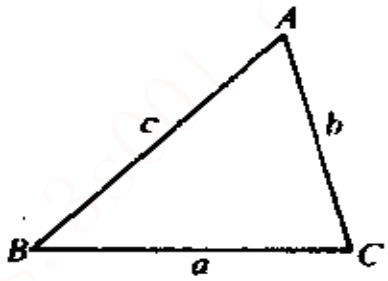


图 6.17(Figure 6.17)

A triangle is a figure with three segments of a straght line connecting in sequence with each other, denoted as the symbol “△”; as is shown in figure 6.17, the triangle is written as “△ABC”, read “triangle ABC”.

在三角形 ABC 中，组成三角形的线段叫做三角形的**边**，如图 6.17 中，线段 AB、BC、CA 是三角形的边. 有时也可以用 a，b，c 表示，如图 6.17 所示，A 点所对的边 BC 用 a 表示，B 点所对的边 AC 用 b 表示，C 点所对的边 AB 用 c 表示.

In triangle ABC, the segments forming a triangle are called the **sides** of the triangle, such as the segments AB, BC and CA in figure 6.17. Sometimes we can use a, b, c to denote sides, such as in figure 6.17, side BC opposite point A denoted as a, side AC opposite point B denoted as b and side AB opposite point C denoted as c.

三角形相邻两边的公共端点叫做三角形的**顶点**，如图 6.17 所

示，A，B，C 是三角形的三个顶点.

A common endpoint between two adjacent sides of a triangle is called a **vertex** of the triangle. For instance, A, B and C in figure 6.17 are three vertices of the triangle.

三角形相邻两边所组成的角叫做三角形的**内角**，简称三角形的**角**，如图 6.17 所示，$\angle A$，$\angle B$，$\angle C$ 是三角形的角.

Angles formed by two adjacent sides of a triangle are called the **interior angles** of a triangle, simplified as the **angles** of a triangle, such as $\angle A$, $\angle B$, $\angle C$ in figure 6.17.

三角形的一边与另一边延长组成的角，叫做三角形的**外角**，如图 6.18 中的 $\angle ACD$.

Angles formed with one side of a triangle extending to another are called the **exterior angles** of a triangle, such as $\angle ACD$ in figure 6.18.

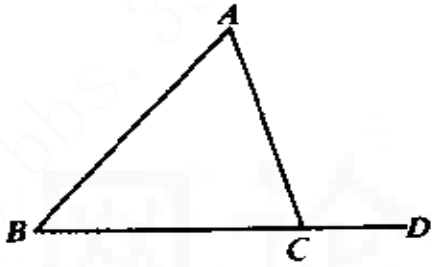


图 6.18(Figure 6.18)

(二) 三条重要线段(Three Important Segments)

1. 三角形的角平分线(Angular Bisectors of a Trangle)

三角形一个角的平分线与这个角的对边相交，这个角的顶点和交点之间的线段叫做三角形的**角平分线**. 如图 6.19，射线 AD 平分 $\angle BAC$，交对边 BC 于点 D，线段 AD 就是 $\triangle ABC$ 的一条角平分线. 则：$\angle BAD = \angle DAC$

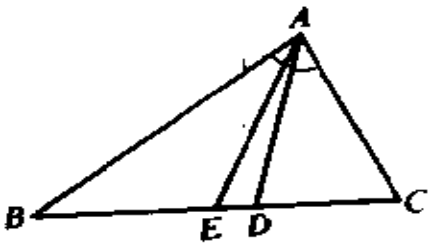


图 6.19(Figure 6.19)

$= \frac{1}{2}\angle BAC$.

The bisector of an angle in a triangle intersects the opposite side of the angle and the segment between the vertex of the angle and the intersecting point is called the **angular bisector** of the triangle. As is shown in figure 6.19, ray AD bisects $\angle BAC$ and intersects the opposite BC at point D. The resulting segment AD is an angular bisector of $\triangle ABC$. So $\angle BAD = \angle DAC = \frac{1}{2}\angle BAC$.

2. 三角形的中线(Central Lines of a Triangle)

连接一个顶点和它的对边中点的线叫做三角形的**中线**. 如图6.19所示,E 是 BC 边的中点,线段 AE 是 $\triangle ABC$ 中的一条中线,有 $BE = EC = \frac{1}{2}BC$.

A line connecting one vertex with its opposite mid-point is called the **central line** of a triangle. As is shown in figure 6.19, E is the mid-point of line BC and segment AE is a mid-point in $\triangle ABC$, getting $BE = EC = \frac{1}{2}BC$.

3. 三角形的高(Altitude of a Triangle)

从三角形一个顶点向它的对边作垂线,顶点和垂足间的线段叫做三角形的**高**. 如图6.20所示,AD, BE, CF 分别为 $\triangle ABC$ 的三条高. 由于 AD 是 $\triangle ABC$ 中 BC 边上的高,则有:$\angle ADB = 90^\circ$, $\angle ADC = 90^\circ$, $\angle ADB = \angle ADC$.

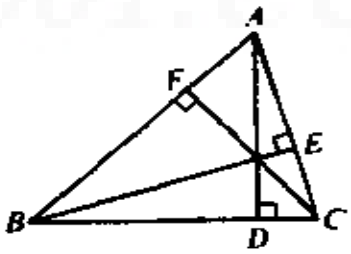


图 6.20(Figure 6.20)

Draw a vertical line from a vertex of a triangle to its opposite side, called base, and the segment between the vertex and the base is called

the **altitude** of the triangle. As is denoted in figure 6.20, AD, BE and CF *are three altitudes of* $\triangle ABC$, respectively. Since AD is the altitude on side BC in $\triangle ABC$, $\angle ADB = 90^\circ$, $\angle ADC = 90^\circ$, $\angle ADB = \angle ADC$.

二、三角形的分类(Classification of Triangles)

(一) 按边分类(in Terms of Sides)

可分为不等边三角形、等腰三角形和等边三角形,如图 6.21 所示.

Trangles can be classified into nonequilateral triangles, isosceles triangles and equilateral triangles, as denoted in figure 6.21.

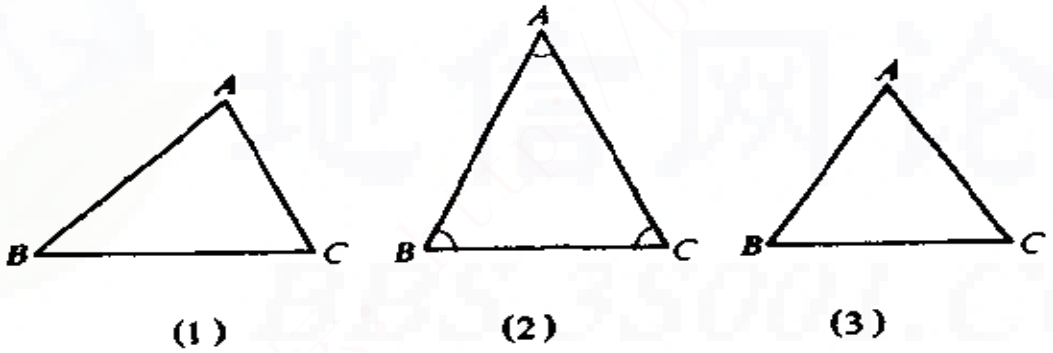


图 6.21(Figure 6.21)

1.不等边三角形(Nonequilateral Triangles)

三角形的三条边都不相等的三角形为**不等边三角形**, 如图 6.21(1)所示.

A triangle in which none of the three sides are equal is called a **nonequilateral triangle**, as shown in figure 6.21(1).

2. 等腰三角形(Isosceles Triangles)

有两条边相等的三角形叫做**等腰三角形**，如图 6.21(2)所示. 在等腰三角形中，相等的两边都叫做腰，另一边叫做**底边**，两腰的夹角叫做**顶角**，腰和底边的夹角叫做**底角**.

A triangle that has two equal sides is called an **isosceles triangle**, as shown in figure 6.21(2). In an isosceles triangle, the two sides which are equal are called **waists**, a third side is called the **baseside**, the included angle between two waists is called a **vertex angle** and the included angles between the waists and the baseside are called **base angles**.

3. 等边三角形(Equilateral Triangles)

三条边都相等的三角形叫做**等边三角形**，它是特殊的等腰三角形，即底边和腰相等的三角形，见图 6.21(3).

A triangle with all three sides equal is called an **equilateral triangle**, which is a special isosceles triangle, that is the triangle with the base side and waists all equal (see figure 6.21(3)).

4. 两个重要性质(Two Important Properties)

1) 三角形任两边的和大于第三边.

The sum of two sides is greater than a third side in a triangle .

2) 三角形任两边的差小于第三边.

The difference of two sides is less than a third side in a triangle.

例 6.1 一个等腰三角形的周长为 18 cm.

(1) 已知腰长是底边长的 2 倍，求各边长.

(2) 已知其中一边长为 4 cm，求其它两边长.

解：

(1) 设底边长为 x，则腰为 $2x$.

$x + 2x + 2x = 18$　　$\therefore$　$x = 3.6$

则，三边分别为 3.6 cm, 7.2 cm, 7.2 cm.

(2) 设长为 4 cm 的边为腰时，则底为 x.

有：$x + 2 \times 4 = 18$　$x = 10$

$\because$　$4 + 4 < 10$，即作为腰的两边之和小于第三边，不可能组成三角形.

$\therefore$　$x = 4$ 只能作为三角形的底.

设 $x = 4$ 为三角形的底，则腰为 x.

$4 + 2x = 18$　$x = 7$

$\therefore$　三角形的三条边分别为 7 cm, 7 cm, 4 cm.

Example 6.1　The perimeter of an isosceles triangle is 18 cm.

(1) Given that the length of a waist is twice the base side, solve for the lengh of each side.

(2) Given that one side is 4 cm, solve for the lenghs of the other two sides.

Solution

(1) Let the base side be x, then the waist length is $2x$.

$x + 2x + 2x = 18.$　　Therefore　$x = 3.6$

Then the three sides are 3.6 cm, 7.2 cm and 7.2 cm, respectively.

(2) Let the 4 cm side be the waist, then the base is x.

Get　$x + 2 \times 4 = 18$,　$x = 10.$

Since $4 + 4 < 10$, that is, the sum of two waist sides is less than a third side, which cannot form a triangle,

the value $x = 4$ can only be the base of the triangle.

Suppose $x = 4$ is the base of the triangle, then the waist is x.

$4 + 2x = 18$　$x = 7.$

Therefore, the three sides of the triangle are 7 cm, 7 cm and 4 cm, respectively.

(二) 按角分类(In Terms of Angles)

可分为锐角、钝角和直角三角形.

Triangles can be divided into acute, obtuse and right triangles.

三角形内角和定理
(Theorem of the Sum of Interior Angles in a Triangle)

三角形的三个内角之和等于 180°.

The sum of the measures in degrees of the interior angles in a triangle is 180°.

根据三角形内角和定理,三角形的三个内角可能都是锐角,也可能有一个是直角或钝角,但不可能有两个直角或钝角.

In terms of the theorem of the sum of the interior angles of a triangle, the three interior angles may be all acute angles, or one angle may be right or obtuse, but it is impossible to have two right or obtuse angles.

1. 锐角三角形(Acute Triangles)

三个角都是锐角的三角形叫做**锐角三角形**,如图 6.22(1).

The triangle with three acute angles is called an **acute triangle**, as shown in Figure 6.22(1).

2. 直角三角形(Right Triangles)

有一个角是直角的三角形叫做**直角三角形**,如图 6.22(2) 所示. 在直角三角形中,夹直角的两边叫做**直角边**,直角的对边叫做**斜边**. 两条直角边相等的直角三角形叫做**等腰直角三角形**. 直角三

角形可以用符号“Rt△”表示，直角三角形 ABC 可以写成“Rt△ABC”，读作“直角三角形 ABC”.

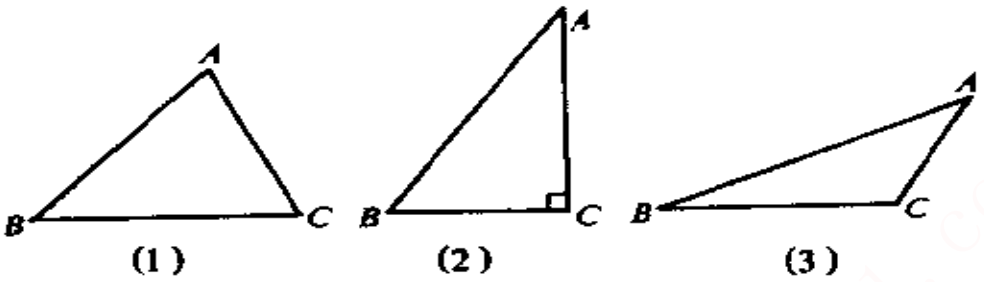


图 6.22(Figure 6.22)

The triangle with one right angle is called a **right triangle**, as is shown in figure 6.22(2). In a right triangle, the two sides of the included right angle are called the **right angled sides** and its opposite side is called the **hypotenuse**. The right triangle with equal right angled sides is called an **isosceles right triangles**. A right triangle may be expressed in a symbol as “Rt△”. Therefore, the right triangle ABC may be written as “Rt△ABC” and read “right triangle ABC”.

3. 钝角三角形(Obtuse Triangles)

有一个角是钝角的三角形叫做**钝角三角形**，如图 6.22(3)所示. 锐角三角形和钝角三角形合称为**斜三角形**.

The triangle with an obtuse angle is called an **obtuse triangle**, as is shown in figure 6.22(3). The acute and obtuse triangles are both referred to as **slant triangles**.

4. 两个重要性质(Two Important Properties)

1) 三角形的一个外角等于和它不相邻的两个内角之和.

The measure in degrees of an exterior angle of a triangle equals the sum of the measures in degrees of its two nonneighboring

interior angles.

2）三角形的一个外角大于任何一个和它不相邻的内角.

An exterior angle of a triangle is greater than either of the two nonneighboring interior angles.

例6.2　已知:如图6.23所示,D是AB上的一点,E是AC上的一点,BE,CD相交于点F,$\angle A = 62^\circ$,$\angle ACD = 35^\circ$,$\angle ABE = 20^\circ$.

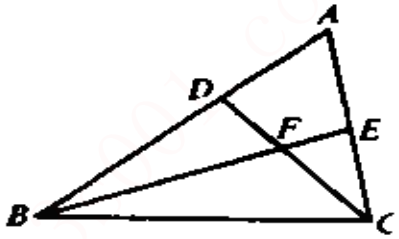


图 6.23(Figure 6.23)

求　(1) $\angle BDC$ 的度数

(2) $\angle BFD$ 的度数

解:(1) $\angle BDC = \angle A + \angle ACD$

$\therefore \quad \angle BDC = 62^\circ + 35^\circ = 97^\circ$.

(2) $\because \quad \angle BFD = 180^\circ - \angle BDC - \angle ABE$

(三角形内角和定理)

$\therefore \quad \angle BFD = 180^\circ - 97^\circ - 20^\circ = 63^\circ$

Example 6.2　Given that ,as shown in figure 6.23, D is a point on line AB , E is a point on line AC, BE and CD are intersected at point F ,

$\angle A = 62^\circ$,$\angle ACD = 35^\circ$,$\angle ABE = 20^\circ$.

Find

(1) the measure in degrees of $\angle BDC$.

(2) the measure in degrees of $\angle BFD$.

Solution

(1) $\angle BDC = \angle A + \angle ACD$

Therefore, $\angle BDC = 62^\circ + 35^\circ = 97^\circ$.

(2) $\angle BFD = 180^0 - \angle BDC - \angle ABE$

(the theorem of the sum of the interior angles of a triangle)

Therefore, $\angle BFD = 180^\circ - 97^\circ - 20^\circ = 63^\circ$

三、特殊三角形的性质
(Properties of Special Triangles)

(一) 等腰三角形(Isosceles triangles)

1. 等腰三角形的两个底角相等. 如图 6.24 中, $\angle B = \angle C$.

The two base angles of an isosceles triangle are equal. As is shown in figure 6.24, $\angle B = \angle C$.

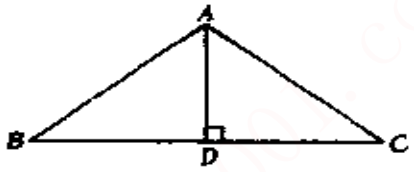


图 6.24(Figure 6.24)

2. 等腰三角形顶角的平分线平分底边并垂直于底边. 如图 6.24, $BD = CD$, $\angle ADB = \angle ADC = 90^\circ$. 即: AD 平分 BC, 且, $AD \perp BC$.

The bisector of the vertex in an isosceles triangle both bisects and is perpendicular to the base side. As is shown in figure 6.24, $BD = CD$, $\angle ADB = \angle ADC = 90^\circ$ .That is, AD bisects BC ,and $AD \perp BC$.

3. 等腰三角形的顶角平分线、底边上的中线、底边上的高互相重合.

In an isosceles triangle, the bisector of the vertex, the mid-line of the base side and the altitude of the base side are coincident with each other.

4. 等边三角形的各角都相等, 且每一个角都等于 60°.

In an equilateral triangle, each of the angles is identical with a

measure in degrees of 60.

5. 有一个角等于 60°的等腰三角形是等边三角形.

The isosceles triangle with an angle equal to 60° is an equilateral triangle.

(二) 直角三角形(Right Triangles)

1. 直角三角形的两个锐角互余. 如图 6.25 中, $\angle A + \angle B = 90^\circ$.

In a right triangle, the two acute angles are complementary to each other. As is shown in figure 6.25, $\angle A + \angle B = 90^\circ$.

2. 在直角三角形中,如果一个锐角等于 30°, 那么它所对的直角边等于斜边的一半. 如图 6.25 所示,若 $\angle A = 30^\circ$, 则, $BC = \frac{1}{2}AB$.

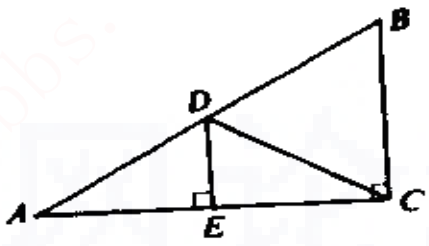


图 6.25(Figure 6.25)

In a right triangle, if one acute angle is 30°, then the length of the leg opposite the 30° angle is half the length of the hypotenuse. As is denoted in figure 6.25, if $\angle A = 30^\circ$, then $BC = \frac{1}{2}AB$.

3. 勾股定理(the Pythagorean Theorem)

任一直角三角形两直角边 a, b 的平方和等于斜边 c 的平方.

In a right triangle, the sum of the squares of the two legs a and b are equal to the square of the hypotenuse c.

$$a^2 + b^2 = c^2$$

4. 勾股定理的逆定理(the Inverse Pythagorean Theorem)

如果三角形的三条边长 a，b，c 有关系 $a^2 + b^2 = c^2$ 成立，则这个三角形是直角三角形.

In a triangle, if the three sides a, b and c have the relationship $a^2 + b^2 = c^2$, then this triangle is a right one.

四、三角形的面积求解(Solution for the Area of Triangles)

1. 已知三角形的一条底边 b 及底边上的高 h(如图 6.26 所示)，则三角形的面积 A 为

$$A = \frac{1}{2}bh$$



(1) (2)

图 6.26(Figure 6.26)

Given one base b and its altitude h (see figure 6.26), then the area A of a triangle is

$$A = \frac{1}{2}bh.$$

2. 已知三角形的两条边长 a，b 及它们的夹角 C 的值(如图 6.27)，则三角形的面积求解公式为

$$A = \frac{1}{2}ab\sin C$$

Given the lengths of two sides a, b and the value of their included angle C (figure 6.27), then the formula for the area of a triangle is

$$A = \frac{1}{2}ab\ \sin C.$$

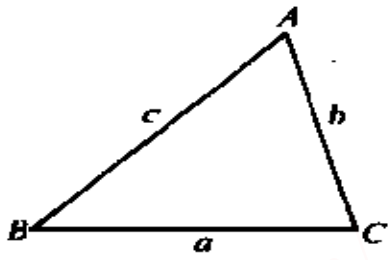


图 6.27(Figure 6.27)

3. 已知三角形三条边 a, b, c，求该三角形的面积，如图 6.27.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

其中，$s = \frac{1}{2}(a+b+c)$ 为 $\frac{1}{2}$ 三角形的周长.

Given the three sides a, b, c of a triangle, solve for the area of the triangle as is denoted in figure 6.27.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is $\frac{1}{2}$ the perimeter of the triangle.

4. 已知三角形的一条边长 c，及相邻两个角 A, B，求三角形的面积，其公式为

$$A = \frac{c^2 \sin A \sin B}{2\sin(A+B)}$$

Given one side c of a triangle and its two adjacent angles A and B, solve for the area of the triangle, of which the formula is

$$A = \frac{c^2 \sin A \sin B}{2\sin(A+B)}$$

第七章　四边形的图形表达（Quadrilateral Graphs）

§7.1　基本概念(Basic Concepts)

一、定义(Definitions)

由不在同一直线上的四条线段首尾顺次相接组成的图形叫做**四边形**.组成四边形的各条线段叫做四边形的**边**,每相邻两条边的公共端点叫做四边形的**顶点**. 四边形用表示它的各个顶点的字母来表示,如图 7.1 中的四边形,可以按照顶点的顺序,记作"四边形 *ABCD*".

The graph with four segments not on the same line connected with one after another is called a **quadrilateral**. Each of the four segments forming a quadrilateral is called its **side** and the common endpoint of every two adjacent sides is called a **vertex** of the quadrilateral. A quadrilateral can be expressed in the letters of its vertices, such as the quadrilateral in figure 7.1, which can be written as "quadrilateral *ABCD*" in sequence of the vertices.



图 7.1(Figure 7.1)

凸四边形(Convex Quadrilaterals)

把四边形的任何一边向两方延长,如果其它各边都在延长所得直线的同一旁,这样的四边形叫做**凸四边形**.如图 7.2 是凸四边形,而图 7.3 则不是凸四边形.

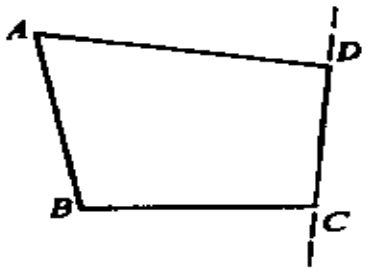


图 7.2(Figure 7.2)

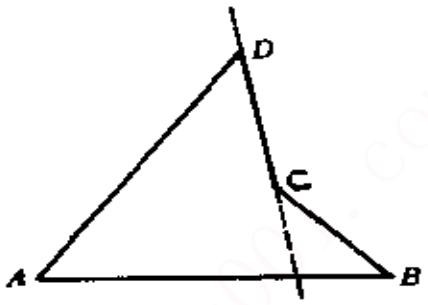


图 7.3(Figure 7.3)

Extend any side of a quadrilateral to both directions and if all the sides are on the same side of the extended line, the resulting quadrilateral is called a **convex quadrilateral**. For example, the figure in figure 7.2 is a convex quadrilateral, while that in figure 7.3 is not a convex quadrilateral.

二、对角线(Diagonals)

在四边形中,连接不相邻两个顶点的线段叫做四边形的**对角线**.如图 7.4 中,在四边形 *ABCD* 中,*AC*、*BD* 是四边形 *ABCD* 的两条对角线.

In a quadrilateral, the line segment that joins two vertices not adjacent to each other is called a **diagonal** of the quadrilateral. As is shown in figure 7.4, *AC* and *BD* are the two diagonal lines of quadrilateral *ABCD* .

三、四边形的内角和外角
(Interior and Exterior Angles of a Quadrilateral)

四边形相邻两边所组成的角叫做四边形的**内角**,简称四边形**的角**.如图 7.5 所示,$\angle B$,$\angle D$ 都是四边形的**内角**.四边形可以用一条对**角**线将其分成两个三角形.在图 7.5 中,四边形 $ABCD$ 可分成为 $\triangle ADC$ 和 $\triangle ABC$.四边形的四个角的和就是这两个三角形**的**内角的**和**,所以有

四边形的内角和等于 360°.

In a quadrilateral, the angles composed of two adjacent sides are called the **interior angles** of the quadrilateral, simplified as the **angles** of a quadrilateral. As is shown in figure 7.5, $\angle B$ and $\angle D$ are both the interior angles of the quadrilateral. A quadrilateral can be divided into two triangles in a diagonal, such that in figure 7.5, quadrilateral $ABCD$ is divided into $\triangle ADC$ and $\triangle ABC$. The sum in degrees of the four angles of a quadrilateral is the sum in degrees of the interior angles of the two triangles, and therefore

the sum in degrees of the interior anlges of a quadrilateral is 360°.

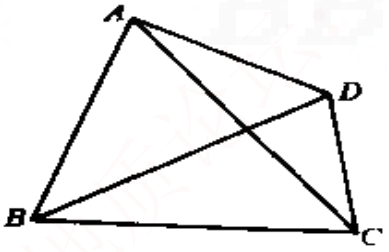


图 7.4(Figure 7.4)

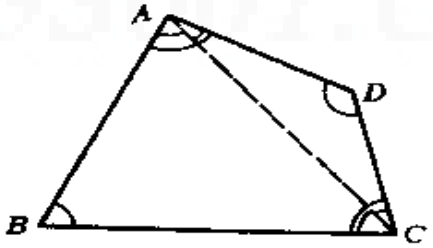


图 7.5(Figure 7.5)

四边形的角的一边与另一边的延长线所组成的角叫做四边形的**外角**,如图 7.6,$\angle\alpha$,$\angle\beta$,$\angle\gamma$,$\angle\delta$ 均为四边形的外角.$\angle 1 + \alpha = 180^\circ$,$\angle 2 + \angle\beta = 180^\circ$,$\angle 3 + \angle\gamma = 180^\circ$.四**边**形的外角是

与它有公共顶点的内角的邻补角.

四边形的外角和等于 360°.

如图 7.6，$\angle\alpha + \angle\beta + \angle\gamma + \angle\delta = 360^\circ$.

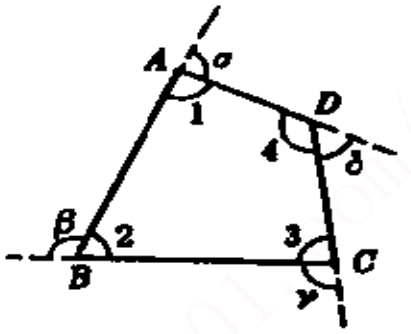


图 7.6(Figure 7.6)

The angle formed by one side of an angle of a quadrilateral and the extension line of the other side is called an **exterior angle** of the quadrilateral, which is supplementary to the interior angle with a common vertex, such that in figure 7.6, $\angle\alpha, \angle\beta, \angle\gamma, \angle\delta$ are all exterior angles of the quadrilateral and $\angle 1 + \alpha = 180^\circ$, $\angle 2 + \angle\beta = 180^\circ$, $\angle 3 + \angle\gamma = 180^\circ$.

The sum in degrees of the exterior anlges of a quadrilateral is 360°. For instance, in figure 7.6, $\angle\alpha + \angle\beta + \angle\gamma + \angle\delta = 360^\circ$.

四、多边形(Polygons)

在平面内，由一些线段首尾相接组成的图形叫做**多边形**，多边形的边、顶点、内角、外角、对角线的意义和四边形相同，多边形有几条边就是几边形．如图 7.7，该多边形有六条边，我们称这个多边形为**六边形**.

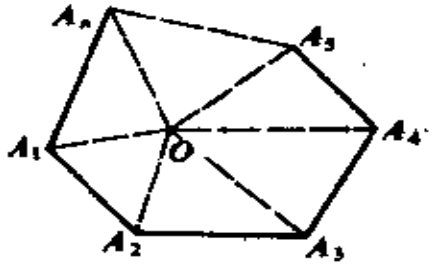


图 7.7(Figure 7.7)

In a plane, the figure composed of several line segments joined at every endpoint in succession is called a **polygon**, whose sides, vertices, interior angles, exterior angles and diagonals are indentical in meaning to those of a quadrilateral. The number of sides of a polygon determines what polygon

it is. For example, the polygon in figure 7.7 has six sides, so we call it a **hexagon**.

多边形的内角和定理(Theorem of the Sum of Interior Angles)

n 边形的内角和等于$(n-2)\cdot 180^\circ$.

The sum in degrees of the interior angles of a polygon of n sides is $(n-2)\cdot 180^\circ$.

多边形的外角和定理(Theorem of the Sum of Exterior Angles)

任意多边形的外角和等于 360°.

The sum in degrees of the exterior angles of any polygon is 360°.

§7.2 平行四边形(Parallelograms)

一、定义(Definitions)

两组对边分别平行的四边形叫做**平行四边形**,如图 7.8 所示,在四边形 $ABCD$ 中,$AB \parallel CD$,$AD \parallel BC$,那么四边形 $ABCD$ 是平行四边形.平行四边形用符号"$\square$"表示,平行四边形 $ABCD$ 记作"$\square ABCD$",读作"平行四边形 $ABCD$".

The quadrilateral with its opposite sides parallel is called a **parallelogram**. As is shown in figure 7.8, in the parallelogram $ABCD$, if $AB \parallel CD$, $AD \parallel BC$, then the quadrilateral $ABCD$ is a parallelogram. A parallelogram can be denoted by the symbol "$\square$", so the parallelogram $ABCD$ is written as "$\square ABCD$", read



图 7.8(Figure 7.8)

"parallelogram $ABCD$".

平行四边形是一种特殊的四边形，如图 7.9 所示，它除有四边形的性质外，还有一些特殊的性质.

A parallelogram is a particular quadrilateral, such that in figure 7.9, in addition to the qualities of a quadrilateral, it also has some special qualities.

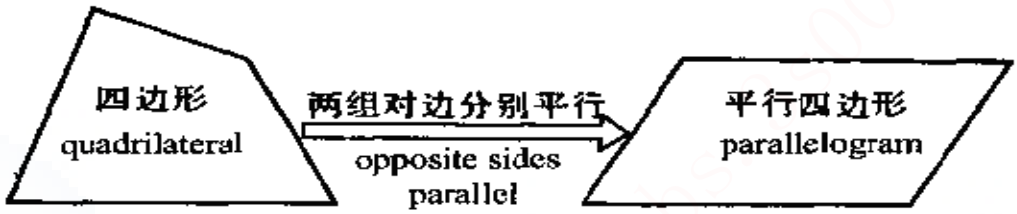


图 7.9(Figure 7.9)

二、平行四边形的性质(Properties of a Parallelogram)

性质 1(Property 1)

平行四边形的对角相等.

The opposite angles of a parallelogram is equal.

性质 2(Property 2)

平行四边形的对边相等.

The opposite sides of a parallelogram is equal.

例 7.1　作 $\square ABCD$ 的对角线 AC，AC 将 $\square$ 分为 $\triangle ABC$ 和 $\triangle CDA$(如图 7.10 所示).

证明：

$AB = CD$　　$CB = AD$　　$\angle B = \angle D$

$\angle BAD = \angle BCD$

解：

$\because$ $AB \parallel CD$, $AD \parallel BC$

$\therefore$ $\angle 1 = \angle 3$, $\angle 2 = \angle 4$

又 $\because$ $AC = CA$

$\therefore$ $\triangle ABC \cong \triangle CDA$

$\therefore$ $AB = CD$ $CB = AD$ $\angle B = \angle D$

又 $\because$ $\angle 1 + \angle 4 = \angle 2 + \angle 3$

$\therefore$ $\angle BAD = \angle BCD$

即： 平行四边形对角相等，平行四边形对边相等.

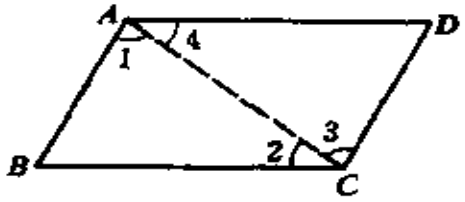


图 7.10(Figure 7.10)

Example 7.1 Given the diagonal AC of $\square ABCD$, AC divides the parallelogram into $\triangle ABC$ and $\triangle CDA$, as shown in figure 7.10.

Illustrate $AB = CD$, $CB = AD$

$\angle B = \angle D$, $\angle BAD = \angle BCD$.

Illustration

Since $AB \parallel CD$, $AD \parallel BC$,

$\angle 1 = \angle 3$, $\angle 2 = \angle 4$.

Also since $AC = CA$, $\triangle ABC \cong \triangle CDA$.

Therefore, $AB = CD$, $CB = AD$, $\angle B = \angle D$.

Similarly, since $\angle 1 + \angle 4 = \angle 2 + \angle 3$, $\angle BAD = \angle BCD$.

It turns out that the opposite angles of a parallelogram are equal and so are the opposite sides of a parallelogram.

性质 3(Property 3)

平行四边形的对角线互相平分.

The diagonals of a parallelogram bisect each other.

例 7.2 如图 7.11 所示，已知：在 $\square ABCD$ 中，对角线 AC、BD 相交

于点 O.

求证：$OA = OC$，$OB = OD$

证明：

∵ 在 $\square ABCD$ 中，$AB \parallel CD$

∴ $\angle 1 = \angle 4$，$\angle 2 = \angle 3$

又

∵ $AB = CD$

（平行四边形的对边相等）

∴ $\triangle OAB \cong \triangle OCD$

∴ $OA = OC$，$OB = OD$

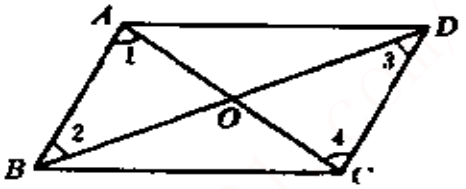


图 7.11(Figure 7.11)

即： 平行四边形的对角线互相平分.

Example 7.2 Given that, as shown in figure 7.11, the diagonals AC and BD are intersected at point O.

Illustrate $OA = OC$, $OB = OD$.

Illustration In $\square ABCD$, $AB \parallel CD$,

thus $\angle 1 = \angle 4$, $\angle 2 = \angle 3$.

Similarly, $AB = CD$

(The opposite sides of a parallelogram are equal)

thus $\triangle OAB \cong \triangle OCD$, $OA = OC$, $OB = OD$.

It turns out that the diagonals of a parallelogram bisect each other.

三、平行四边形的判定(Determination of a Parallelogram)

判定定理 1(Determination 1)

两组对角分别相等的四边形是平行四边形.

The quadrilateral in which two pairs of opposite angles are equal, respectively, is a parallelogram.

判定定理 2(Determination 2)

两组对边分别相等的四边形是平行四边形.

The quadrilateral in which two pairs of opposite sides are equal, respectively, is a parallelogram.

判定定理 3(Determination 3)

对角线互相平分的四边形是平行四边形.

The quadrilateral in which two diagonals bisect each other is a parallelogram.

判定定理 4(Determination 4)

一组对边平行且相等的四边形是平行四边形.

The quadrilateral in which a pair of opposite sides are both parallel and equal is a parallelogram.

例 7.3 已知:如图 7.12,四边形 $ABCD$ 的对角线 AC、BD 相交于点 O,并且 $AO = CO$, $BO = DO$.

求证: 四边形 $ABCD$ 是平行四边形.

证明:

$\because$ $AO = CO$, $BO = DO$,

$\angle 1 = \angle 2$

$\therefore$ $\triangle AOB \cong \triangle COD$

$AB = CD$

同理,$AD = BC$

$\therefore$ 四边形 $ABCD$ 是平行四边形

(两组对边相等的四边形是平行四边形)

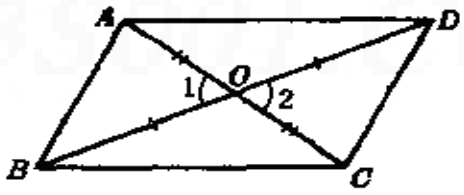


图 7.12(Figure 7.12)

Example 7.3 Given, as shown in figure 7.12, that diagonals AC and BD of a quadrilateral are intersected at point O, and $AO = CO$, $BO = DO$.

Illustrate that the quadrilateral $ABCD$ is a parallelogram.

Illustration

$AO = CO$, $BO = DO$, $\angle 1 = \angle 2$.

Thus $\triangle AOB \cong \triangle COD$, $AB = CD$.

Similarly, $AD = BC$.

So the quadrilateral $ABCD$ is a parallelogram. (The quadrilateral in which two pairs of opposite sides are equal is a parallelogram.)

例 7.4　如图7.13所示,已知在四边形 $ABCD$ 中,$AB \parallel CD$, $AB = CD$.

求证：　四边形 $ABCD$ 是平行四边形.

证明：　连结 AC.

$\because$　$AB \parallel CD$

$\therefore$　$\angle 1 = \angle 2$

$\because$　$AB = CD$, $AC = CA$

$\therefore$　$\triangle ABC \cong \triangle CDA$

$\therefore$　$BC = AD$

$\therefore$　四边形 $ABCD$ 是平行四边形(两组对边分别相等的四边形是平行四边形)

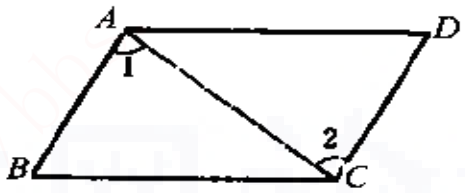


图 7.13(Figure 7.13)

Example 7.4　Given that, in quadrilateral $ABCD$, $AB \parallel CD$, $AB = CD$ (see figure 7.13)

Illustrate　Quadrilateral $ABCD$ is a parallelogram.

Illustration　Join AC.

From $AB \parallel CD$　　get $\angle 1 = \angle 2$.

From $AB = CD$, $AC = CA$,　　get $\triangle ABC \cong \triangle CDA$.

Therefore, $BC = AD$.

Thus, the quadrilateral $ABCD$ is a parallelogram.

(The quadrilateral in which two pairs of opposite sides are equal, respectively, is a parallelogram.)

§7.3 矩形、菱形、正方形
(Rectangles, Lozenges and Squares)

矩形、菱形和正方形都是特殊的平行四边形.

Rectangles, lozenges or rhombuses and squares are all special parallelograms.

一、矩形(Rectangles)

有一个角是直角的平行四边形叫做**矩形**,通常也叫长方形,如图 7.14 所示.矩形是一种特殊的平行四边形.

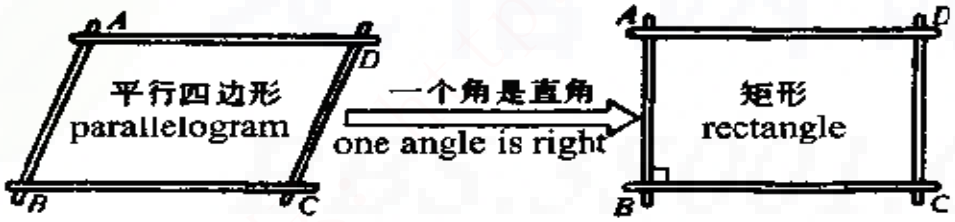


图 7.14(Figure 7.14)

The parallelogram with one angle a right angle is called a **rectangle**, as is shown in figure 7.14. A rectangle is a special parallelogram.

(一) 矩形的性质(Properties of Rectangles)

矩形性质定理 1(Property 1)

矩形的四个角都是直角.

The four angles of a rectangle are all right angles.

矩形性质定理 2(Property 2)

矩形的对角线相等.

The diagonals of a rectangle are equal.

例 7.5　已知,矩形 $ABCD$(如图 7.15)

求证:　$AC = DB$

证明:　在矩形 $ABCD$ 中,

$\because$　$\angle ABC = \angle DCB = 90^\circ$

(矩形的4个角都是直角)

$AB = DC$,　$BC = CB$

$\therefore$　$\triangle ABC \cong \triangle DCB$

$\therefore$　$AC = DB$

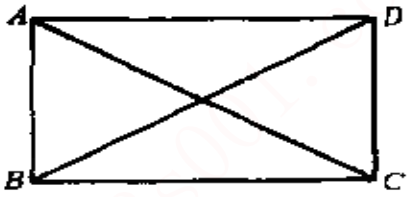


图 7.15(Figure 7.15)

Example 7.5　Given rectangle $ABCD$ (see figure 7.15)

Illustrate　$AC = DB$.

Illustration　In rectangle $ABCD$,

$\angle ABC = \angle DCB = 90^\circ$.

(The four angles of a rectangle are all right angles.)

Thus $AB = DC$,　$BC = CB$.

So $\triangle ABC \cong \triangle DCB$ and $AC = DB$.

(二) 矩形的判定(Determination of Rectangles)

矩形的判定定理 1(Determination 1)

有三个角是直角的四边形是矩形.

The quadrilateral with three angles right angles is a rectangle.

矩形的判定定理 2(Determination 2)

对角线相等的平行四边形是矩形.

The parallelogram with equal diagonals is a rectangle.

例 7.6　已知，在 $\square ABCD$ 中，

$AC = DB$（如图 7.16）.

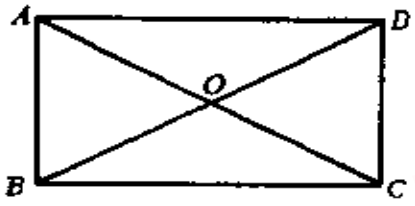


图 7.16(Figure 7.16)

求证：$\square ABCD$ 是矩形

证明：

$\because$ $AC = DB$, $BC = CB$,

$AB = DC$

$\therefore$ $\triangle ABC \cong \triangle DCB$

$\therefore$ $\angle ABC = \angle DCB$

又 $\because$ $AB \parallel DC$

$\therefore$ $\angle ABC + \angle DCB = 180^\circ$ $\therefore$ $\angle ABC = 90^\circ$

$\therefore$ $\square ABCD$ 是矩形

（有一个角是直角的平行四边形是矩形）

Example 7.6

Given that, in $\square ABCD$,

$AC = DB$ (figure 7.16), illustrate $\square ABCD$ is a rectangle.

Illustration

Since $AC = DB$, $BC = CB$, $AB = DC$,

$\triangle ABC \cong \triangle DCB$ $\angle ABC = \angle DCB$.

Also since $AB \parallel DC$, $\angle ABC + \angle DCB = 180^\circ$,

$\angle ABC = 90^\circ$.

Thus $\square ABCD$ is a rectangle. (The parallelogram with one angle right one is a rectangle.)

二、菱形(Lozenges or Rhombuses)

有一组邻边相等的平行四边形叫做**菱形**，如图 7.17 所示. 菱形也是一种特殊的平行四边形.

The parallelogram with a pair of adjacent sides equal is called a lozenge or rhombus, as is shown in figure 7.17. A lozenge is also a

special parallelogram.

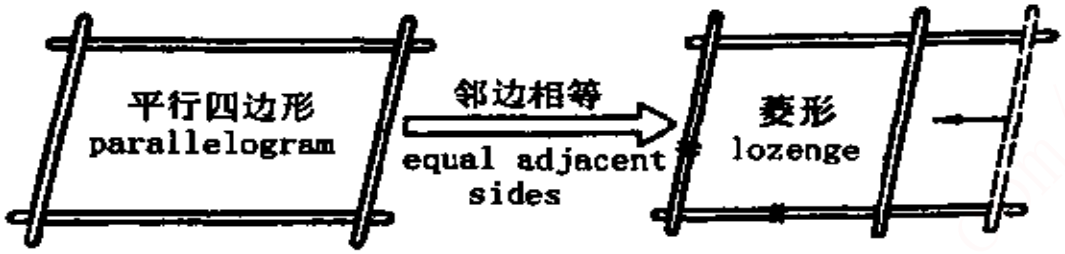


图 7.17(Figure 7.17)

(一) 菱形的性质(Properties of Lozenges)

菱形性质定理 1(Property 1)

菱形的四条边都相等.

All the four sides of a lozenge are equal.

菱形性质定理 2(Property 2)

菱形的对角线互相垂直,并且每一条对角线平分一组对角.

The diagonals of a lozenge are perpendicular to each other, each diagonal bisecting a pair of opposite angles.

例 7.7 如图 7.18 所示,已知,在菱形 $ABCD$ 中,对角线 AC 和 BD 相交于点 O.

求证:$AC \perp BD$,AC 平分 $\angle BAD$ 和 $\angle BCD$;

BD 平分 $\angle ABC$ 和 $\angle ADC$.

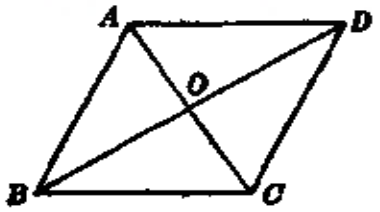


图 7.18(Figure 7.18)

证明: $\because$ 四边形 $ABCD$ 是菱形

$\therefore$ $AB = AD$ (菱形的四条边都相等)

在等腰 $\triangle ABD$ 中,

$\because$ $BO = OD$

$\therefore$ $AC \perp BD$，AC 平分 $\angle BAD$

同理，AC 平分 $\angle BCD$，BD 平分 $\angle ABC$ 和 $\angle ADC$.

Example 7.7 Given that, in lozenge $ABCD$, diagonals AC and BD intersect at O (see figure 7.18), illustrate $AC \perp BD$, AC bisects $\angle BAD$ and $\angle BCD$; BD bisects $\angle ABC$ and $\angle ADC$.

Illustration Since quadrilateral $ABCD$ is a lozenge,

$AB = AD$ (All the four sides of a lozenge are equal.)

In isosceles triangle ABD, $BO = OD$,

thus $AC \perp BD$, AC bisects $\angle BAD$.

Similarly, AC bisects $\angle BCD$; BD bisects $\angle ABC$ and $\angle ADC$.

(二) 菱形的判定(Determination of Lozenges)

菱形的判定定理 1(Determination 1)

四边都相等的四边形是菱形.

The quadrilateral with four sides all equal is a lozenge.

菱形的判定定理 2(Determination 2)

对角线互相垂直的平行四边形是菱形.

The parallelogram with diagonals perpendicular to each other is a lozenge.

例 7.8 已知，$\square ABCD$ 中，对角线 $AC \perp BD$ (如图 7.19).

求证：$\square ABCD$ 是菱形

证明：$\because$ 四边形 $ABCD$ 是平行四边形

$\therefore$ $OB = OD$

又 $\because$ $AC \perp BD$，垂足为 O

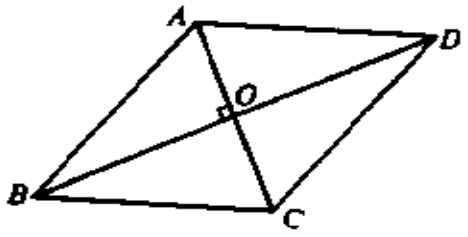


图 7.19(Figure 7.19)

$\therefore$ $AB = AD$ $\square ABCD$ 是菱形.

Example 7.8 Given that, in $\square ABCD$, diagonals $AC \perp BD$ (see figure 7.19), illustrate $\square ABCD$ is a lozenge.

Illustration

Since quadrilateral $ABCD$ is a parallelogram,

$OB = OD$.

Also since $AC \perp BD$ with the vertical base O,

$AB = AD$.

Thus $\square ABCD$ is a lozenge.

三、正方形(Squares)

有一组邻边相等并且有一个角是直角的平行四边形叫做**正方形**.由此,正方形既是有一组邻边相等的矩形,又是有一个角是直角的菱形(如图 7.20). 所以正方形具有矩形的性质,同时又具有菱形的性质.

The parallelogram with a pair of adjacent sides equal and one angle a right angle is called a **square**. Accordingly, a square is not only a rectangle with a pair of adjacent sides equal but a lozenge with an angle a right angle as well (see figure 7.20). Therefore, a square has the qualities of both a rectangle and a lozenge.

正方形性质定理 1(Property 1)

正方形的四个角都是直角,四条边都相等.

All the four angles of a square are right angles and all its four sides are equal.

正方形性质定理 2(Property 2)

正方形的两条对角线相等,并且互相垂直平分,每条对角线平

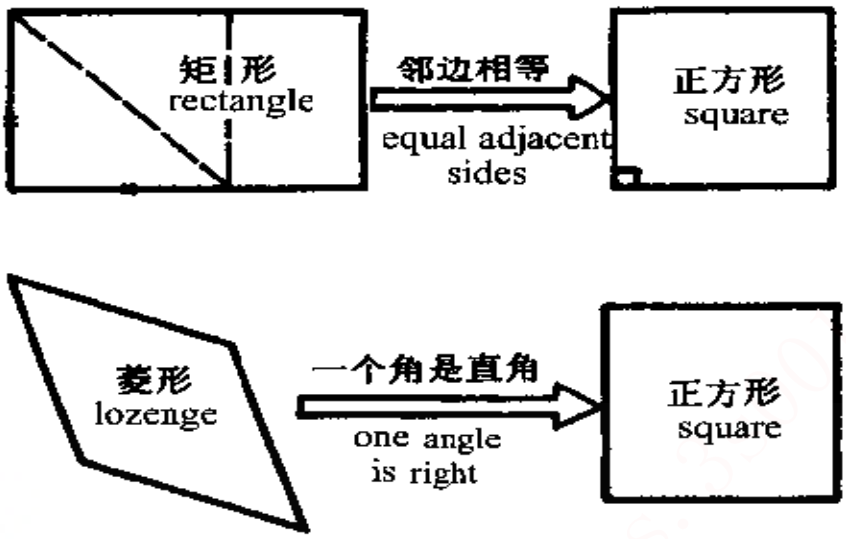


图 7.20(Figure 7.20)

分一组对角.

The two diagonals of a square are not only equal but vertical to and bisect each other, as well, each diagonal bisecting a pair of opposite angles.

例 7.9　求证正方形的两条对角线把正方形分成四个全等的等腰直角三角形.

已知:四边形 $ABCD$ 是正方形,对角线 AC、BD 相交于点 O(如图 7.21).

求证:$\triangle ABO$,$\triangle BCO$,$\triangle CDO$,$\triangle DAO$ 是全等的等腰直角三角形.

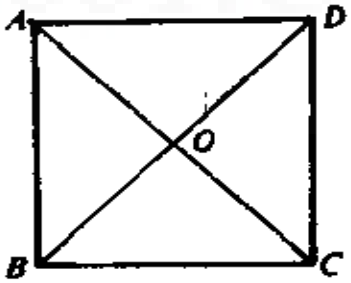


图 7.21(Figure 7.21)

证明：

∵ 四边形 $ABCD$ 是正方形

∴ $AC = BD$，$AC \perp BD$，

$AO = CO = BO = DO$

（正方形的两条对角线相等，并且互相垂直平分）

∴ $\triangle ABO$，$\triangle BCO$，$\triangle CDO$，$\triangle DAO$ 都是等腰直角三角形，并且 $\triangle ABO \cong \triangle BCO \cong \triangle CDO \cong \triangle DAO$

Example 7.9 Illustrate that the two diagonals of a square divide the square into four congruent isosceles right triangles.

Given that the quadrilateral $ABCD$ is a square with its diagonals AC and BD intersecting at point O (see figure 7.21), illustrate $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle DAO$ are four congruent isosceles right triangles.

Illustration Since quadrilateral $ABCD$ is a square,

$AC = BD$, $AC \perp BD$, $AO = CO = BO = DO$.

(The two diagonals of a square are equal and vertical to and bisect each other.)

Thus $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle DAO$ are all isosceles right triangles and $\triangle ABO \cong \triangle BCO \cong \triangle CDO \cong \triangle DAO$.

矩形、菱形、正方形都是平行四边形，但它们都是有特殊条件的平行四边形．正方形不仅是特殊的平行四边形，而且是邻边相等的特殊矩形，也是有一个角是直角的特殊菱形．

Rectangles, lozenges and squares are all parallelograms, but they are all parallelograms in special conditions. A square is not only a special parallelogram, but a special rectangle with adjacent sides equal and a special lozenge with one angle a right angle, as well.

§7.4 梯形(Trapezoids)

两组对边分别平行的四边形是平行四边形,而只有一组对边平行的另一种特殊四边形就是梯形(见图 7.22).

The quadrilateral in which two pairs of opposite sides are parallel, respectively, is a parallelogram, but there is another special quadrilateral in which only a pair of opposite sides are parallel to each other. This quadrilateral is called a trapezoid (see figure 7.22).



图 7.22(Figure 7.22)

一、定义(Definitions)

一组对边平行而另一组对边不平行的四边形叫**做梯形**(如图 7.23). 平行的两边叫做**梯形的底**(通常把较短的底叫做上底,较长的底叫做下底),不平行的两边叫**做梯形的腰**,两底的距离叫做**梯形的高**. 如图 7.23 所示的梯形中, AD 叫做上底, BC 叫做下底, AB、DC 为腰, EF 为高.

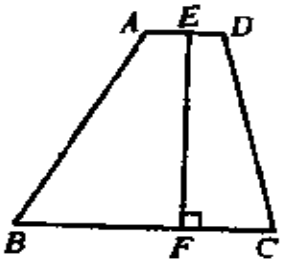


图 7.23(Figure 7.23)

The quadrilateral in which one pair of opposite sides are parallel and the other are not parallel is called a **trapezoid** (figure 7.23). The parallel sides are the **bases of the trapezoid** (usually the short one is called the upper base and the long one the bottom base); the two sides which are not parallel are called the **waists of the trapezoid** and the perpendicular distance between the bases is called its **altitude**. As is shown in figure 7.23, in the trapezoid, AD is the upper base, BC the bottom one, both AB and DC are waists, and EF the altitude.

二、特殊梯形及其性质
(Special Trapezoids and Their Properties)

一腰垂直于底的梯形叫做直角梯形(如图 7.24),两腰相等的梯形叫做等腰梯形(如图 7.25)

The trapezoid with one waist perpendicular to a base is called a right-angled trapezoid (see figure 7.24); the trapezoid with two waists equal is called an isosceles trapezoid (see figure 7.25)

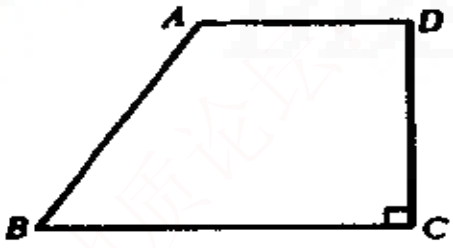


图 7.24(Figure 7.24)

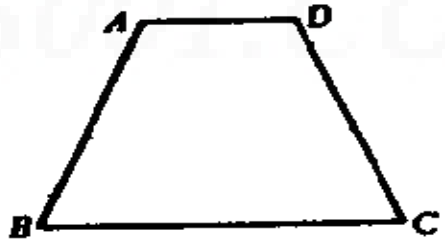


图 7.25(Figure 7.25)

直角梯形和等腰梯形都是特殊梯形(如图 7.26).

Right-angled trapezoids and isosceles trapezoids are both special trapezoids (see figure 7.26).

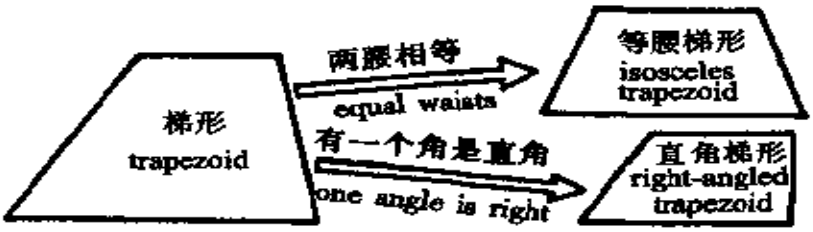


图 7.26(Figure 7.26)

等腰梯形性质定理 1(Property 1)

等腰梯形在同一底上的两个角相等.

In an isosceles trapezoid, two angles on the same base are equal.

例 7.10　已知,如图 7.27,在梯形 $ABCD$ 中,$AD \parallel BC$, $AB = DC$.

求证:$\angle B = \angle C$

证明:过点 D 作 $DE \parallel AB$,交 BC 于点 E,得到 $\triangle DEC$.

$\because$ $AD \parallel BC$, $DE \parallel AB$

$\therefore$ $AB = DE$

$\because$ $AB = DC$

$\therefore$ $DE = DC$, $\angle 1 = \angle C$

$\because$ $\angle 1 = \angle B$　$\therefore$ $\angle B = \angle C$

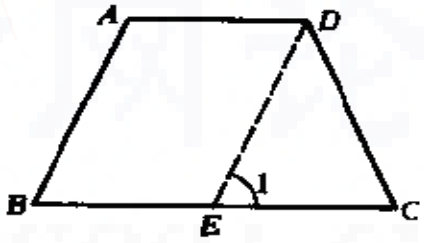


图 7.27(Figure 7.27)

Example 7.10　Given that, in trapezoid $ABCD$ (figure 7.27), $AD \parallel BC$, $AB = DC$, illustrate $\angle B = \angle C$.

Illustration　Make $DE \parallel AB$ from point D, intersecting BC at point E and obtaining $\triangle DEC$.

Since $AD \parallel BC$, $DE \parallel AB$, get $AB = DE$.

Since $AB = DC$, get $DE = DC$ and $\angle 1 = \angle C$.

Also since $\angle 1 = \angle B$, $\angle B = \angle C$.

等腰梯形性质定理 2(Property 2)

等腰梯形的两条对角线相等.

In an isosceles trapezoid, two diagonals are equal.

例 7.11　已知,在梯形 $ABCD$ 中,$AD \parallel BC$,$AB = DC$(见图 7.28)

求证:　$AC = DB$

证明:　在梯形 $ABCD$ 中,

$\because$　$AB = DC$

$\therefore$　$\angle ABC = \angle DCB$

(等腰梯形在同一底上的两个角相等)

又　$\because$　$BC = CB$

$\therefore$　$\triangle ABC \cong \triangle DCB$

$\therefore$　$AC = DB$

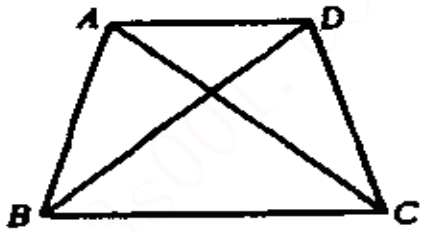


图 7.28(Figure 7.28)

Example 7.11　Given that, in trapezoid $ABCD$, $AD \parallel BC$, $AB = DC$ (see figure 7.28), illustrate $AC = DB$.

Illustration　In trapezoid $ABCD$, since $AB = DC$,

$\angle ABC = \angle DCB$.

(In a isosceles trapezoid, two angles on the same base are equal.)

Also since $BC = CB$, $\triangle ABC \cong \triangle DCB$.

Thus $AC = DB$.

三、等腰梯形的判定(Determination of Trapezoids)

等腰梯形的判定定理 1(Determination 1)

在同一底上的两个角相等的梯形是等腰梯形.

The trapezoid with two angles equal on the same base is called an isosceles trapezoid.

等腰梯形的判定定理 2(Determination 2)

对角线相等的梯形是等腰梯形.

The trapezoid with its diagonals equal is called an isosceles trapezoid.

例 7.12　已知,在梯形 $ABCD$ 中,$AD \parallel BC$,$AC = BD$ (图 7.29).

求证:$AB = DC$.

证明:过点 D 作 $DE \parallel AC$,交 BC 的延长线于 E,得 $\square ACED$,所以 $DE = AC$.

$\because \quad AC = BD \quad \therefore \quad DE = BD$

$\therefore \quad \angle 1 = \angle E$

$\because \quad \angle 2 \doteq \angle E \quad \therefore \quad \angle 1 = \angle 2$

又　$\because \quad AC = DB,\ BC = CB$

$\therefore \quad \triangle ABC \cong \triangle DCB \quad \therefore \quad AB = DC$

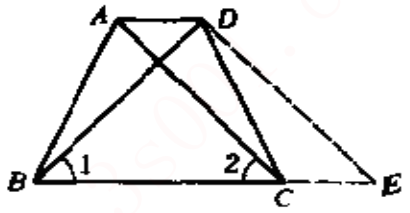


图 7.29(Figure 7.29)

Example7.12　Given that, in trapezoid $ABCD$, $AD \parallel BC$, $AC = BD$ (see figure 7.29). illustrate $AB = DC$.

Illustration　Make $DE \parallel AC$ through point D, intersecting the extended line at point E to get $\square ACED$ and thus $DE = AC$.

Since $AC = BD$, get $DE = BD$.

Thus $\angle 1 = \angle E$.

Since $\angle 2 = \angle E$, get $\angle 1 = \angle 2$.

Also since $AC = DB$ and $BC = CB$, get $\triangle ABC \cong \triangle DCB$.

Thus $AB = DC$.

§7.5 特殊四边形的周长和面积求解公式 (Formulas for Perimeters and Areas of Special Quadrilaterals)

一、正方形(Squares)

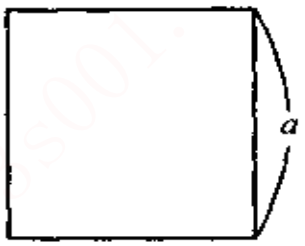


图 7.30(Figure 7.30)

如图 7.30,设正方形边长为 a,

则:周长 $C = 4a$,面积 $A = a^2$.

As is shown in figure 7.30, let the length of a square be a, then the perimeter $C = 4a$ and the area $A = a^2$.

二、矩形(Rectangles)

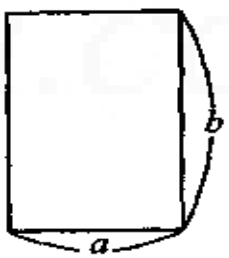


图 7.31(Figure 7.31)

如图 7.31,设矩形短边为 a,长边为 b,则:

矩形周长 $C = 2(a + b)$

矩形面积 $A = ab$

As is shown in figure 7.31, let the short side of a rectangle be a and the long side b, then the perimeter of the rectangle $C = 2(a + b)$ and its area $A = ab$.

三、平行四边行(Parallelograms)

如图7.32,设平行四边形两对边边长分别为 a,b,高为 h,其

中一对邻边夹角为 φ(锐角)，则：平行四边形的周长为 $C = 2(a + b)$；面积为 $S = bh$ 或 $S = ab\sin\varphi$.

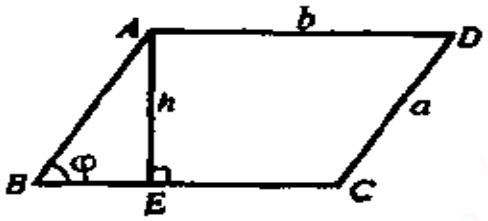


图 7.32(Figure 7.32)

As is shown in figure 7.32, let the lengths of the two pairs of the opposite sides of a parallelogram be a and b, respectively, the altitude h and one pair of the adjacent included angles φ (acute angle), then the perimeter of the parallelogram $C = 2(a + b)$; its area $S = bh$ or $S = ab\sin\varphi$.

四、菱形(Lozenges)

如图 7.33，设菱形边长为 a，高为 h，锐角为 φ 的两对角线分别为 d_1, d_2.

则：菱形周长为 $C = 4a$，面积为

(1) $A = ah$

(2) $A = a^2\sin\varphi$

(3) $A = \dfrac{d_1 d_2}{2}$

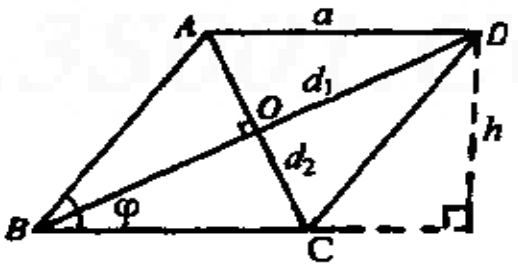


图 7.33(Figure 7.33)

As is shown in figure 7.33, let the length of a side of a lozenge be a, the altitude h, the two pairs of the diagonals with the acute angle φ are d_1 and d_2, respectively.

Then, the perimeter of the lozenge $C = 4a$ and its area:

(1) $A = ah$ (2) $A = a^2\sin\varphi$

(3) $A = \frac{d_1 d_2}{2}$

五、梯形(Trapezoids)

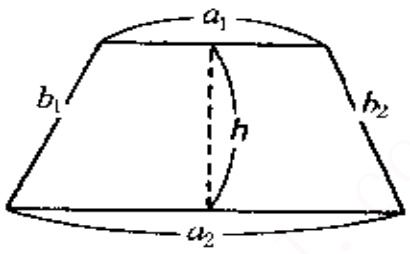


图 7.34(Figure 7.34)

如图 7.34,设梯形上底为 a_1,下底为 a_2,两腰分别为 b_1, b_2,高为 h.则:

周长 $C = a_1 + b_1 + a_2 + b_2$

面积 $A = \frac{a_1 + a_2}{2} \cdot h$

As is shown in figure 7.34, let the upper base of a trapezoid be a_1, the bottom base a_2, the two waists are b_1 and b_2, respectively, the altitude h. Then,

perimeter $C = a_1 + b_1 + a_2 + b_2$,

area $A = \frac{a_1 + a_2}{2} \cdot h$.

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第八章　圆的图形表达(Circle Graphs)

§8.1　基本概念(Basic Concepts)

一、定义(Definitions)

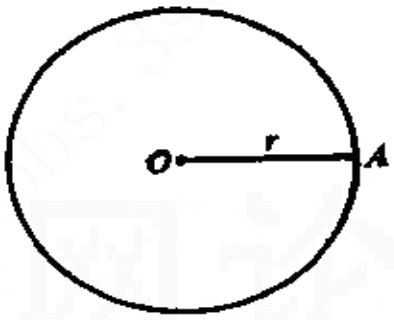


图 8.1(Figure 8.1)

如图 8.1 所示，在一个平面内，线段 OA 绕它固定的端点 O 旋转一周，另一个端点 A 随之旋转所形成的图形叫做**圆**，固定的端点 O 叫做**圆心**，线段 OA 叫做**半径**，以点 O 为圆心的圆，记作"$\odot O$"，读作"圆 O".

As is shown in figure 8.1, in a plane, the graph in which line segment OA revolves about its fixed endpoint O to 360° is called a **circle**. The fixed endpoint O is called the **center** and the segment OA is called the **radius**. The circle with point O the center is written as "$\odot O$", read "circle O".

从上面定义知道(From the definitions above, we can know)：

(1) 圆上各点到定点(圆心 O) 的距离都等于定长(半径 r).

The distance from each point of a circle to a fixed point (center O) equals a given length (called radius r).

(2) 到定点的距离等于定长的点都在圆上.

All points at a given distance are on the circle.

因此，圆是到定点的距离等于定长的点的集合；而圆的内部可以看作是到圆心的距离小于半径的点的集合（图 8.2 中的 P 点）；圆的外部可以看作是到圆心的距离大于半径的点的集合（图 8.2 中的 Q 点）.

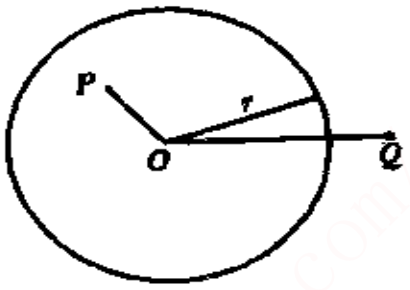


图 8.2(Figure 8.2)

Therefore, a circle is the set of all points at a given distance from a fixed point in a plane; the inside of a circle can be regarded as the set of all points (point P in figure 8.2) from which the distance to the center are less than the radius; the outside of a circle can be regarded as the set of all points (point Q in figure 8.2) from which the distance to the center are greater than the radius.

二、圆的弦和弧(Chords and Arcs of Circles)

(一) 弦(Chords)

连接圆上任意两点的线段叫做**弦**（如图 8.3 中的 CD），经过圆心的弦（如图 8.3 中的 AB）叫做**直径**，直径等于半径的 2 倍.

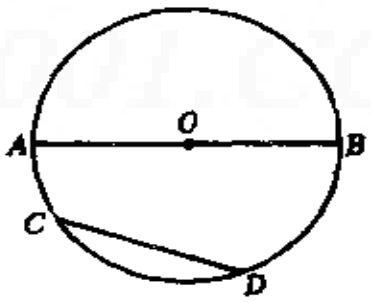


图 8.3(Figure 8.3)

The segment joining any two points on a circle is called a **chord** of the circle such as segment CD in figure 8.3. The chord passing through the center of the circle (AB in figure 8.3) is called the **diameter**, which is twice the radius.

(二) 弧(Arcs)

圆上任意两点间的部分叫做**圆弧**,简称**弧**.弧用符号"⌒"表示,以 A,B 为端点的弧记作$\overset{\frown}{AB}$,读作"圆弧 AB"或"弧 AB"

The curve bounded by two given points of a circle is called an **arc of circle**, simplified as an **arc** which is expressed in the symbol "⌒". The arc at endpoints A and B is written as $\overset{\frown}{AB}$, read "arc of circle AB" or "arc AB"

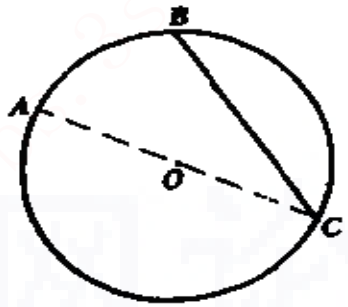


图 8.4(Figure 8.4)

圆的任意一条直径的两个端点分圆成两条弧,每一条弧都叫做**半圆**.大于半圆的弧(用三个字母表示,如图 8.4 中的$\overset{\frown}{BAC}$)叫做**优弧**;小于半圆的弧(用两个字母表示,如图 8.4 中的$\overset{\frown}{BC}$)叫做**劣弧**.由弦及其所对的弧组成的图形叫做**弓形**.

A diameter at any two endpoints divides a circle into two arcs, either of which is called a **half-circle**. The arc greater than a half-circle (expressed in three letters such as $\overset{\frown}{BAC}$ in figure 8.4) is called a **superior arc** and that less than a half-circle (denoted by two letters, such as $\overset{\frown}{BC}$ in figure 8.4) is called an **inferior arc**. The figure consisting of a chord and its opposite arc is called a **segment of a circle**.

三、同心圆,等圆(Concentric Circles, Equal Circles)

(一) 同心圆(Concentric Circles)

圆心相同,半径不相等的两个圆叫做**同心圆**.如图8.5中的两

个圆是以点 O 为圆心的同心圆.

The two circles having a common center and unequal radii are called **concentric circles**, such as the two circles in figure 8.5 which are the concentric circles with point O the center.

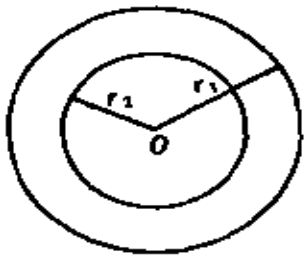


图 8.5(Figure 8.5)



图 8.6(Figure 8.6)

(二) 等圆(Equal Circles)

能够重合的两个圆叫做**等圆**.半径相等的两个圆是等圆.如图 8.6 所示,$\odot O_1$ 和 $\odot O_2$ 的半径都等于 r,因此它们是两个等圆.

Two circles which can be coincident with each other are called **equal circles** or two circles having equal radii are called equal circles. As is shown in figure 8.6, the radii of $\odot O_1$ and $\odot O_2$ are both equal to r, so they are two equal circles.

同圆或等圆的半径相等.在同圆和等圆中,能够互相重合的弧叫做**等弧**.

The radii of the same or equal circles are equal. In the same or equal cirlces, the arcs which are coincident with each other are called **equal arcs**.

例 8.1　已知:如图 8.7,在 $\odot O$ 中,AB、CD 为直径.

求证:$AD \parallel BC$

证明:　连结 AC、BD, AB、CD 为 $\odot O$ 的直径.

则　$OA = OB$, $OC = OD$

∴　四边形 $ADBC$ 为平行四边形.

∴　$AD \parallel BC$

Example 8.1

Given that, in $\odot O$ in figure 8.7, AB and CD are diameters, illustrate $AD \parallel BC$.

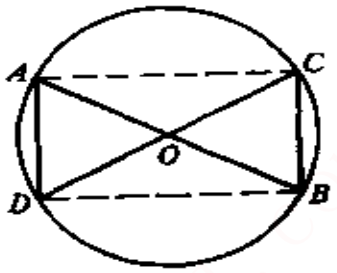


图 8.7(Figure 8.7)

Illustration　Join AC and BD.

AB and CD are the diameter of $\odot O$, then $OA = OB$, $OC = OD$.

Therefore, quadrilateral $ADBC$ is a parallelogram and AD is parallel to BC.

四、过三点的圆(A Circle Passing through Three Points)

定理(Theorem)

不在同一直线上的三个点确定一个圆,过同一条直线上的三点不能作圆.

A circle can be determined by three points not on one line, and a circle cannot be made by three points passing through one line.

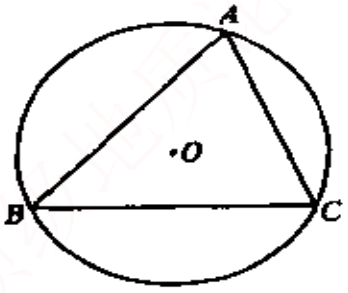


图 8.8(Figure 8.8)

由定理可知,经过三角形三个顶点可以作一个圆.经过三角形各顶点的圆叫做三角形的**外接圆**,外接圆的圆心叫做三角形的**外心**,这个三角形叫做这个圆的**内接三角形**. 如图8.8所示,$\odot O$ 是 $\triangle ABC$ 的外接圆,圆心 O 是 $\triangle ABC$ 的外心,$\triangle ABC$ 是 $\odot O$ 的内接三角形.

From the theorem above, a circle can be made by passing through

three vertices of a triangle. The circle passing through three vertices of a triangle is called a **circumcircle** of the triangle whose center is called the **circumcenter** of the triangle and this triangle is called an **inscribed triangle.** As is shown in figure 8.8, circle O is the circumcircle of triangle ABC whose center O is the circumcenter and triangle ABC is the inscribed triangle of circle O .

五、圆心角、弦心矩(Central Angles and Chord-Center Moments)

(一) 圆心角(Central Angles)

顶点在圆心的角称为**圆心角**,如图 8.9 所示角 α.

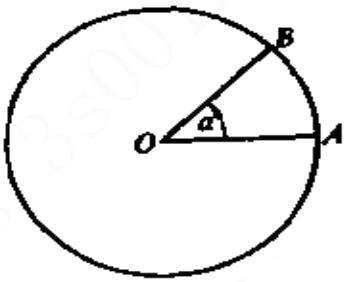


图 8.9(Figure 8.9)

The angle with its vertex at the center of a circle is called a **central angle**, such as the angle α in figure 8.9.

(二) 弦心矩(Chord-Center Moments)

从圆心到弦的距离叫做**弦心矩**,如图 8.10 中的 OM.

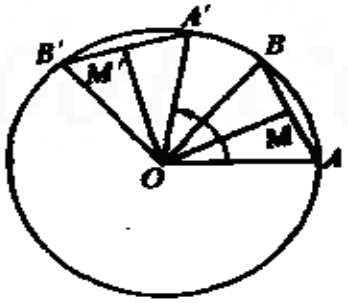


图 8.10(Figure 8.10)

The distance from the center of a circle to the chord is called a **chord-center Moment**, such as OM in figure 8.10.

在同圆或等圆中,相等的**圆心角**所对的弧相等,所对的弦相等,所对的弦的弦心矩相等,如图 8.10 中的 OM 与 OM'.

In the same or equal circles, the opposite arcs having equal central angles are equal and so are the opposite chords and the moments of chord

and center, such as OM and OM' in figure 8.10.

例 8.2　如图 8.11，已知 AB 和 CD 为 $\odot O$ 的两条直径，弦 $CE \parallel AB$，$\overset{\frown}{EC}$ 为 40°，求 $\angle BOD$ 的度数.

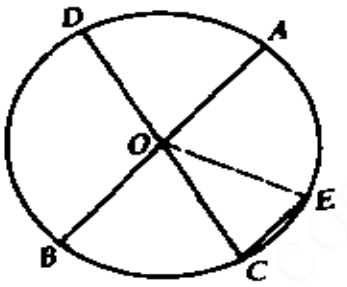


图 8.11(Figure 8.11)

解：连结 OE，

$\because$ $\overset{\frown}{EC}$ 的度数为 40°

$\therefore$ $\angle COE = 40^\circ$，$\angle C = 70^\circ$

$\because$ $AB \parallel EC$

$\therefore$ $\angle BOC = \angle C = 70^\circ$

$\therefore$ $\angle BOD = 110^\circ$

Example 8.2　In figure 8.11, given that AB and CD are two diameters of circle O, chord CE is parallel to chord AB and $\overset{\frown}{EC}$ is 40°, find the degree of angle BOD.

Solution　Join OE,

Since $\overset{\frown}{EC}$ in degree is 40°, $\angle COE = 40^\circ$, $\angle C = 70^\circ$.

Since $AB \parallel EC$, $\angle BOC = \angle C = 70^\circ$.

Therefore, $\angle BOD = 110^\circ$.

六、圆周角(Circumferential Angles)

顶点在圆上，并且两边都和圆相交的角叫做**圆周角**.

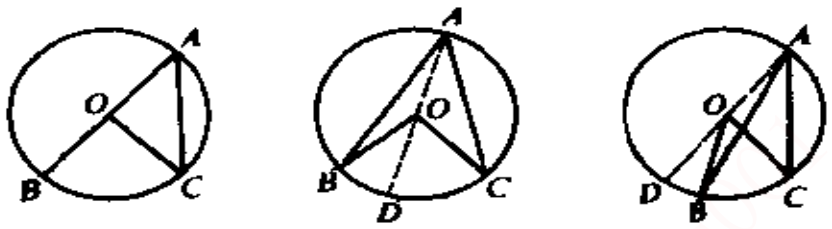
The angle whose vertex is on a circle with both sides intersecting the circle is called a **circumferential angle**.

(一) 圆周角与圆心角的关系(Relationship between a Circumferential Angle and a Central Angle)

一条弧所对的圆周角等于它所对的圆心角的一半. 如图 8.12

所示，$\overset{\frown}{BC}$ 所对的圆心角为 $\angle BOC$，圆周角为 $\angle BAC$.

The circumferential angle opposite an arc equals half its opposite central angle. For example, in figure 8.12, the central angle opposite $\overset{\frown}{BC}$ is $\angle BOC$ and the circumferential angle is $\angle BAC$.



(1) (2) (3)

图 8.12(Figure 8.12)

1. 图 8.12(1) 中，圆心 O 在 $\angle BAC$ 的一条边上.

In figure 8.12(1), the center O is on one side of angle BAC.

$$OA = OC,\ \angle C = \angle BAC,\ \angle BOC = \angle BAC + \angle C,$$

$$\therefore\quad \angle BAC = \frac{1}{2}\angle BOC.$$

2. 图 8.12(2) 中，圆心 O 在 $\angle BAC$ 的内部，作直径 AD，利用 1 的结果，有

In figure 8.12(2), the center O is inside angle BAC. Make radius AD and use the result of 1 to get

$$\angle BAD = \frac{1}{2}\angle BOD,\ \angle DAC = \frac{1}{2}\angle DOC$$

$$\therefore\quad \angle BAD + \angle DAC = \frac{1}{2}(\angle BOD + \angle DOC)$$

$$\therefore\quad \angle BAC = \frac{1}{2}\angle BOC$$

3. 图 8.12(3) 中，圆心 O 在 $\angle BAC$ 的外部，作直径 AD，利用 1 的结果有

In figure 8.12(3), the center O is outside angle BAC . Make radius AD and use the result of 1 to get

$$\angle DAB = \frac{1}{2}\angle DOB,\ \angle DAC = \frac{1}{2}\angle DOC$$

$$\therefore\quad \angle DAC - \angle DAB = \frac{1}{2}(\angle DOC - \angle DOB)$$

$$\therefore\quad \angle BAC = \frac{1}{2}\angle BOC$$

例 8.3　如图 8.13，OA、OB、OC 都是 $\odot O$ 的半径，

$$\angle AOB = 2\angle BOC$$

求证：$\angle ACB = 2\angle BAC$

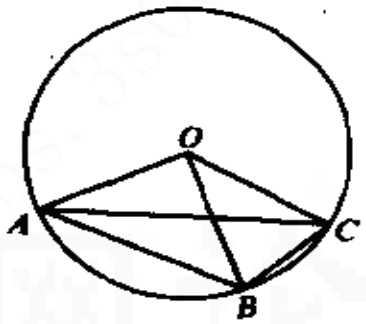


图 8.13(Figure 8.13)

证明：

$$\angle ACB = \frac{1}{2}\angle AOB$$

$$\angle BAC = \frac{1}{2}\angle BOC$$

$$\angle AOB = 2\angle BOC \quad \therefore\quad \angle ACB = 2\angle BAC$$

Example 8.3　As is shown in figure 8.13, OA、OB、OC are all the radii of circle O , $\angle AOB = 2\angle BOC$.

Illustrate $\angle ACB = 2\angle BAC$.

Illustration　$\angle ACB = \frac{1}{2}\angle AOB$,

$$\angle BAC = \frac{1}{2}\angle BOC,\ \angle AOB = 2\angle BOC.$$

Therefore, $\angle ACB = 2\angle BAC$.

(二) 推论(Deduction)

1. 同弧或等弧所对的圆周角相等；同圆或等圆中，相等的圆周角

所对的弧也相等，如图 8.14 所示，同一条弧 $\overset{\frown}{AB}$ 所对的角 $\angle AC_1B = \angle AC_2B = \angle AC_3B$. Circumferential angles opposite the same or equal arcs are equal; in the same or equal circles, the arcs opposite equal circumferential angles are equal, too. In the case of figure 8.14, the angles opposite the same arc $\overset{\frown}{AB}$ are

$$\angle AC_1B = \angle AC_2B = \angle AC_3B.$$

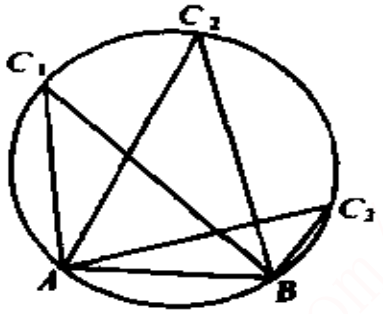


图 8.14(Figure 8.14)

2. 半圆(或直径)所对的圆周角是直角；90°的圆周角所对的弦是直径，如图 8.15 所示

$\angle AC_1B = \angle AC_2B = \angle AC_3B = 90^\circ$

The circumferential angle opposite a half-circle or a diameter is a right angle; the chord opposite a circumferential angle of 90° is a diameter, such as in figure 8.15.

$\angle AC_1B = \angle AC_2B = \angle AC_3B = 90^\circ$

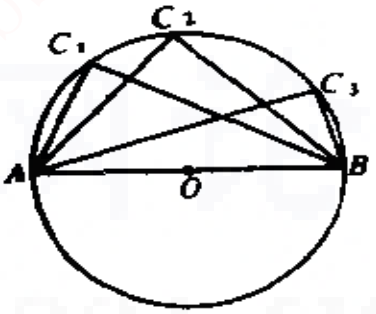


图 8.15(Figure 8.15)

例 8.4　如图 8.16，在 $\odot O$ 中，直径 AB 为 10 cm，弦 AC 为 6 cm，$\angle ACB$ 的平分线交 $\odot O$ 于 D，求 BC、AD 和 BD 的长.

解：　∵　AB 为直径　∴　$\angle ACB = \angle ADB = 90^\circ$

在 Rt△ACB 中，

$$BC = \sqrt{AB^2 - AC^2} = \sqrt{10^2 - 6^2} = 8\ (\text{cm})$$

∵　CD 平分 $\angle ACB$

∴　$\overset{\frown}{AD} = \overset{\frown}{BD}$　∴　$AD = BD$

在等腰直角三角形 ADB 中，

$$AD = BD = \frac{\sqrt{2}}{2}AB$$

$$= \frac{\sqrt{2}}{2} \times 10 = 5\sqrt{2}\ (\text{cm})$$

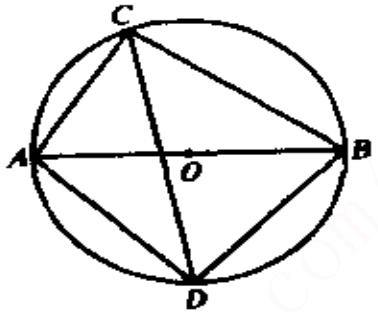


图 8.16(Figure 8.16)

Example 8.4 In figure 8.16, given that, in circle O, diameter AB is 10 cm, chord AC is 6 cm, and the bisector of angle ACB intersects circle O at point D, find the lengths of BC, AD and BD.

Solution Since AB is the diameter, $\angle ACB = \angle ADB = 90^\circ$.

In right triangle ACB,

$$BC = \sqrt{AB^2 - AC^2} = \sqrt{10^2 - 6^2} = 8\ (\text{cm}).$$

Since CD bisects angle ACB, $\overset{\frown}{AD} = \overset{\frown}{BD}$.

Therefore, $AD = BD$.

In the isosceles right triangle ADB,

$$AD = BD = \frac{\sqrt{2}}{2}AB = \frac{\sqrt{2}}{2} \times 10 = 5\sqrt{2}\ (\text{cm}).$$

七、圆的内接四边形(Inscribed Quadrilaterals)

如果一个四边形的所有顶点都在同一个圆上，这个四边形叫做圆内接四边形，这个圆叫做这个四边形的外接圆，如图 8.17 所示.

If all the vertices of a quadrilateral are in the same circle, the quadrilateral is called an **inscribed quadrilateral** of a circle and the circle is called a **circumscribed circle** of the quadrilateral, such as in figure 8.17.

定理(Theorem)

圆的内接四边形的对角互补,并且任何一个外角都等于它的内对角.

The opposite angles of an inscribed quadrilateral of a circle are supplementary and any exterior angle equals its interior opposite angle.

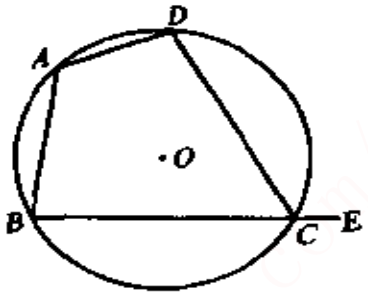


图 8.17(Figure 8.17)

例 8.5 如图 8.18, $\odot O_1$ 和 $\odot O_2$ 都经过 A、B 两点,经过点 A 的直线 CD 与 $\odot O_1$ 交于点 C,与 $\odot O_2$ 交于点 D. 经过点 B 的直线 EF 与 $\odot O_1$ 交于点 E,与 $\odot O_2$ 交于点 F.

求证: $CE \parallel DF$.

证明: 连结 AB.

$\because$ $ABEC$ 是 $\odot O_1$ 的内接四边形

$\therefore$ $\angle BAD = \angle E$.

又

$\because$ $ADFB$ 是 $\odot O_2$ 的内接四边形

$\therefore$ $\angle BAD + \angle F = 180^\circ$.

$\therefore$ $\angle E + \angle F = 180^\circ$

$\therefore$ $CE \parallel DF$

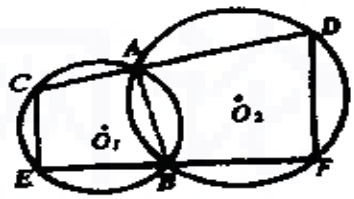


图 8.18(Figure 8.18)

Example 8.5 As is shown in figure 8.18, both circle O_1 and circle O_2 cross points A and B; line CD passing through point A intersects circle O_1 at point C and intersects circle O_2 at point D. Line EF passing through point B intersects circle O_1 at point E and intersects circle O_2 at point F.

Illustrate $CE \parallel DF$.

Illustration Join AB .

Since $ABEC$ is an inscribed quadrilateral of circle O_1,

$\angle BAD = \angle E$.

Also since $ADFB$ is an inscribed quadrilateral of circle O_2,

$\angle BAD + \angle F = 180^\circ$. And $\angle E + \angle F = 180^\circ$.

Therefore, $CE \parallel DF$.

§8.2 直线和圆及圆和圆的位置关系 (Relationship of Position between a Line and a Circle and between One Circle and Another)

一、直线和圆的位置关系 (Relationship of Position between a Line and a Circle)

直线与圆有三种不同的位置关系(如图 8.19 所示).

There are three different relationships of position between a line and a circle (see figure 8.19).

1. 直线和圆有两个公共点时,叫做直线和圆**相交**(如图 8.19(1)),这时直线叫做圆的**割线**.

If there are two common points between a line and a circle, the line and the circle are said to be **intersected** (see figure 8.19(1)), where the line is called a **secant** of the circle.

2. 直线和圆有唯一公共点时,叫做直线和圆**相切**(如图 8.19(2)),这时直线叫做圆**的切线**,唯一的公共点叫做**切点**.

If there is only one common point between a line and a circle, the line is **tangent** to the circle (see figure 8.19(2)), where the line is called a **tangent line** of the circle and the common point is called a

tangent point.

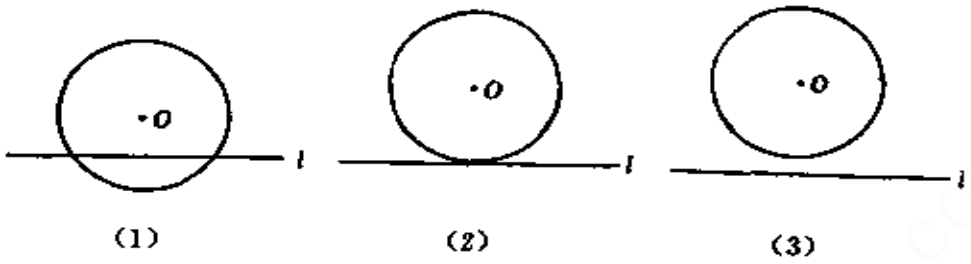


图 8.19(Figure 8.19)

3. 直线和圆没有公共点时,叫做直线和圆**相离**(如图 8.19(3)).

If there is no common point between a line and a circle, the line and the circle are separated (see figure 8.19(3)).

根据直线与圆相交、相切、相离的定义,容易看出,如果 $\odot O$ 的半径为 r,圆心 O 到直线 l 的距离为 d,那么

In terms of the definitions of intersection, tangency and separation between a line and a circle, it is easy to find that if the radius of a circle is r and the distance from center O to line l is d, then

(1) 直线 l 和 $\odot O$ 相交,则 $d < r$.

If line l and circle O are intersected, then $d < r$.

(2) 直线 l 和 $\odot O$ 相切,则 $d = r$.

If line l and circle O are tangent, then $d = r$.

(3) 直线 l 和 $\odot O$ 相离,则 $d > r$.

If line l and circle O are separated, then $d > r$.

例 8.6 在直角三角形 ABC 中,$\angle C = 90^\circ$,$AC = 3$ cm,$BC = 4$ cm,以 C 为圆心,r 为半径的圆与 AB 有怎样的位置关系?为什么?

(1) $r = 2$ cm　(2) $r = 2.4$ cm　(3) $r = 3$ cm

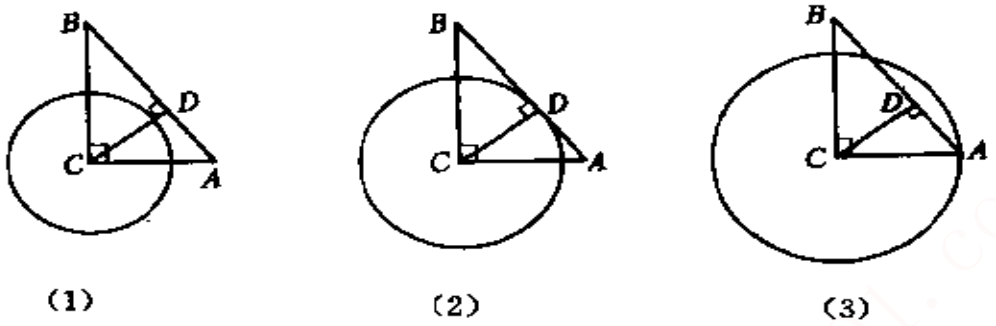


图 8.20(Figure 8.20)

解：　过 C 作 $CD \perp AB$，垂足为 D(图 8.20)，在直角三角形 ABC 中，

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{3^2 + 4^2} = 5$$

根据三角形面积公式有

$$CD \cdot AB = AC \cdot BC$$

$$\therefore \quad CD = \frac{AC \cdot BC}{AB} = \frac{3 \times 4}{5} = 2.4\ (\text{cm})$$

即圆心 C 到 AB 的距离 $d = 2.4$ cm.

(1) 当 $r = 2$ cm 时(如图 8.20(1))，$d > r$，$\odot C$ 与直线 AB 相离.

(2) 当 $r = 2.4$ cm 时(如图 8.20(2))，有 $d = r$，$\odot C$ 与直线相切.

(3) 当 $r = 3$ cm 时(如图 8.20(3))，有 $d < r$，$\odot C$ 和直线 AB 相交.

Example 8.6　In right angle ABC, $\angle C = 90^\circ$, $AC = 3$ cm, $BC = 4$ cm and C is the center. What about the relationship of position between the circle at radius r and line AB? Why?

(1) $r = 2$ cm, (2) $r = 2.4$ cm, (3) $r = 3$ cm.

Solution Make $CD \perp AB$ through point C with D the perpendicular base (see figure 8.20). In right triangle ABC,

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{3^2 + 4^2} = 5.$$

In terms of the area formula of a triangle, get

$$CD \cdot AB = AC \cdot BC.$$

Thus $$CD = \frac{AC \cdot BC}{AB} = \frac{3 \times 4}{5} = 2.4 \text{ (cm)}.$$

That is, the distance from center C to line AB is $d = 2.4$ cm.

(1) When $r = 2$ cm, $d > r$, and circle C and line AB are separated (see figure 8.20(1))

(2) When $r = 2.4$ cm, $d = r$, and circle C and line AB are tangent (see figure 8.20(2)).

(3) When $r = 3$ cm, $d < r$, and circle C and line AB are intersected (see figure 8.20(3)).

二、切线及切线的性质
(Tangent Lines and Properties)

(一) 圆的切线
(Tangent Lines of a Circle)

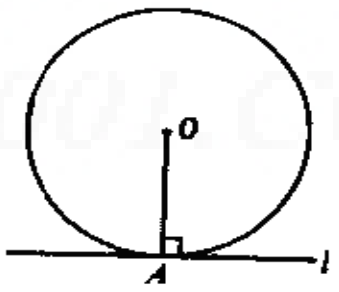


图 8.21(Figure 8.21)

经过半径的外端并且垂直于这条半径的直线是圆的切线. 在 $\odot O$ 中(如图 8.21),经过半径 OA 的外端点 A,作直线 $l \perp OA$,则圆心 O 到直线 l 的距离等于半径 r,则直线 l 与圆一定相切.

The line passing through the outside end of a radius and is perpendicular to the radius is a tangent line of a circle. As is shown in figure 8.21, in circle O, make line l perpendicular to OA through the

outside endpoint A of the radius OA , and then the distance from circle O to line l equals radius r , so that line l must be tangent to the circle.

例 8.7　已知,直线 AB 经过 $\odot O$ 上的点 C,并且 $OA = OB, CA = CB$(图 8.22).

求证:直线 AB 是 $\odot O$ 的切线.

证明:　连接 OC.

$\because$　$OA = OB$, $CA = CB$

$\therefore$　OC 是等腰三角形 OAB 底边 AB 上的中线

$\therefore$　$AB \perp OC$

(直线 AB 经过半径 OC 的外端 C,并且垂直于半径 OC,所以 AB 是 $\odot O$ 的切线)

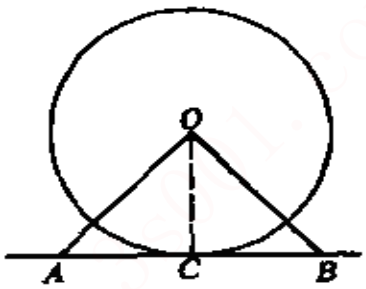


图 8.22(Figure 8.22)

Example 8.7　Given that line AB passes through point C of circle O , and $OA = OB, CA = CB$ (see figure 8.22), illustrate that line AB is a tangent line of circle O .

Illustration　Join OC.

Since $OA = OB$, $CA = CB$, OC is a mid-line on the base side AB of isosceles triangle OAB ,

So　$AB \perp OC$.

Since line AB passes through the outside endpoint C of radius OC and is perpendicular to the radius OC, AB is the tangent line of circle O .

(二) 切线的性质(Properties of Tangent Lines)

圆的切线垂直于经过切点的半径.

A tangent line of a circle is perpendicular to the radius through the tangent point.

推论 1(Deduction 1)

经过圆心且垂直于切线的直线必经过切点.

The line passing through the center of a circle and perpendicular to a tangent line is bound to pass through the tangent point.

推论 2(Deduction 2)

经过切点且垂直于切线的直线必经过圆心.

The line passing through the tangent point and perpendicular to the tangent line is bound to pass through the center of a circle.

三、三角形的内切圆(The Inscribed Circle of a Triangle)

与三角形各边都相切的圆叫做三角形的**内切圆**，内切圆的圆心叫做三角形的**内心**，这个三角形叫做**圆的外切三角形**(如图 8.23).

The circle tangent to all the sides of a triangle is called the **inscribed circle** of a triangle whose center is called the **incenter** of the triangle, and the triangle is called the **circumscribed triangle** of the circle (see figure 8.23).

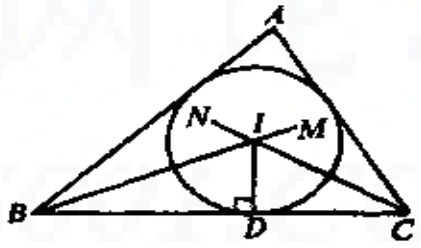


图 8.23(Figure 8.23)

和多边形的各边都相切的圆叫做多边形的**内切圆**. 这个多边形叫做圆的**外切多边形**.

The circle tangent to the sides of a polygon is called the **inscribed circle** of a polygon. This polygon is called the **circumscribed polygon** of the circle.

四、圆和圆的位置关系
(Relationship of Position between Two Circles)

在平面内,两圆相对运动可以得到以下几种不同的位置关系(图 8.24).

In a plane, the relative movement of two circles may bring about the following different relationships (see figure 8.24)

(一) 两圆外离(Outer Separation)

两个圆没有公共点,并且每个圆上的点都在另一个圆的外部时,叫做这两个圆外离(图 8.24(1)).

If there is no common point between two circles and the points of one cirlce is outside the other, these two circles are said to be separated (see figure 8.24(1)).

(二) 两圆外切(Outer Tangency)

两个圆有唯一的公共点,并且除了这个公共点以外,每个圆上的点都在另一个圆的外部时,叫做这两个圆外切(图 8.24(2)),这个唯一的公共点叫做切点.

If two circles have a common point, and besides the common point, the points of one circle are outside the other, we say that these two circles are circumscribed (see figure 8.24(2)) and the only common point is called a tangent point.

(三) 两圆相交(Intersection)

两个圆有两个公共点时,叫做这两个圆相交(图 8.24(3)).

If two circles have two common points, we say that the two circles are intersected (see figure 8.24(3)).

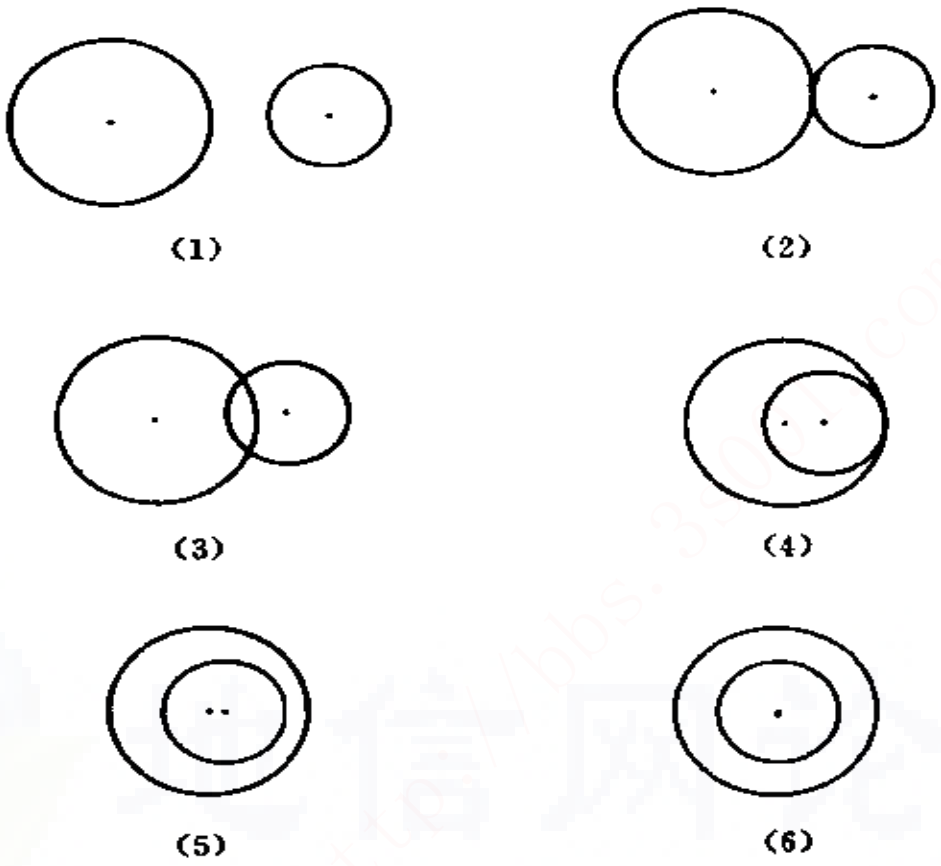


图 8.24(Figure 8.24)

(四) 两圆内切(Inner Tangency)

两个圆有唯一的公共点,并且除了这个公共点以外,一个圆上的点都在另一个圆的内部时,叫做这两个圆内切(图 8.24(4)),这个唯一的公共点叫做切点.

两个圆外切和内切统称为两个圆相切.

Two circles have only one common point, and besides this common point, the points of a circle are all inside the other circle, we say that these two circles are inscribed (see figure 6.24(4)) and this common point is called the tangent point.

Both being circumscribed and inscribed for two circles are called two

circles being tangent.

(五) 两圆内含(Inclusion)

两个圆没有公共点,并且一个圆上的点都在另一个圆的内部时,叫做这两个圆内含(图 8.24(5)). 两圆同心是两圆内含的一种特例(图 8.24(6).

Two circles have no common point and the points of one circle are inside the other circle, we say that these two circles are inclusive (figure 8.24(5)). It is a special case of the inclusion for two circles to have a common center (see figure 8.24(6)).

当两圆的半径一定时,两圆的位置关系与两圆圆心的距离(圆心距)的大小有关,设两圆半径分别为 R 和 r,圆心矩为 d,那么

(1) 两圆外离,则 $d > R + r$.

(2) 两圆外切,则 $d = R + r$.

(3) 两圆相交,则 $R - r < d < R + r$ $(R \geqslant r)$.

(4) 两圆内切,则 $d = R - r$ $(R > r)$.

(5) 两圆内含,则 $d < R - r$ $(R > r)$.

When two cirlces have fixed radii, the relationship of position between two circles is relevant to the distance between the centers of two circles (central distance). Let the radii of two circles be R and r, respectively, and their central distance be d, then

(1) If two circles are separated, then $d > R + r$.

(2) If two circles are circumscribed, then $d = R + r$.

(3) If two circles are intersected,
then $R - r < d < R + r$ $(R \geqslant r)$.

(4) If two circles are inscribed, then $d = R - r$ $(R > r)$.

(5) If two circles are inclusive, then $d < R - r$ $(R > r)$.

例 8.8　如图 8.25，⊙O 的半径为 5 cm，点 P 是 ⊙O 外一点，$OP = 8$ cm.

求：(1) 以 P 为圆心作 ⊙P 与 ⊙O 外切，小圆 ⊙P 的半径是多少？

(2) 以 P 为圆心作 ⊙P 与 ⊙O 内切，大圆 ⊙P 的半径是多少？

解：

(1) 设 ⊙O 与 ⊙P 外切于点 A，

则 $PA = OP - OA$

$\therefore \quad PA = 3$ cm

(2) 设 ⊙O 与 ⊙P 内切于点 B，则

$PB = OP + OB$

$\therefore \quad PB = 13$ cm

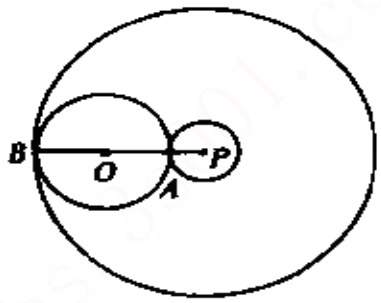


图 8.25(Figure 8.25)

Example 8.8　In figure 8.25, the radius of circle O is 5 cm, point P is a point outside circle O,

$OP = 8$ cm.

Find　(1) the radius of the small circle P by making circle P and circle O circumscribed with P the center.

(2) the radius of the large circle P by making circle P and circle O inscribed with P the center.

Solution

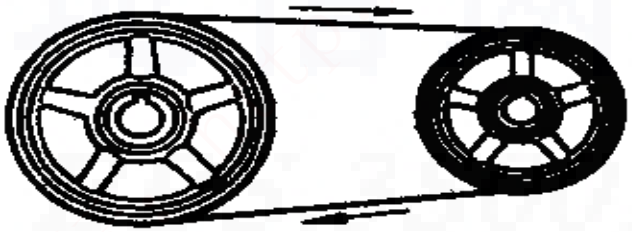
(1) Let circle O and circle P be circumscribed at point A, then $PA = OP - OA$. Therefore, $PA = 3$ cm.

(2) Let circle O and circle P be inscribed at point B, then $PB = OP + OB$. Therefore, $PB = 13$ cm.

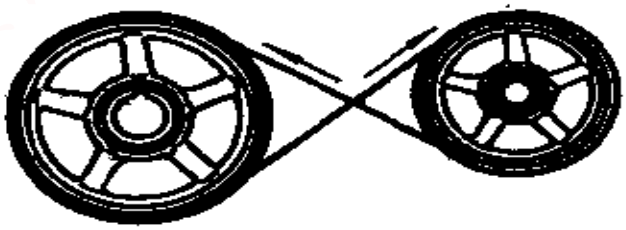
五、两圆的公切线(Common Tangent Lines of Two Circles)

和两个圆都相切的直线,叫做两圆的**公切线**.两个圆在公切线同旁时,这样的公切线叫做**外公切线**(图 8.26(1));两个圆在公切线两旁时,这样的公切线叫做**内公切线**(图 8.26(2)).公切线的两个切点的距离叫做公切线的**长**.

Lines tangent to two circles are called the **common tangents** of the two circles. When two circles are on the same side of their common tangent lines, the tangent lines are called **exterior common tangent lines** (figure 8.26(1)); when two circles are on the two sides of their common tangents, the tangent lines are called **interior common tangent lines** (figure 8.26(2)). The distance between two tangent poins of a common tangent is called the **length** of the common tangent line.



(1)



(2)

图 8.26(Figure 8.26)

例 8.9　如图 8.27，已知 $\odot O_1$、$\odot O_2$ 的半径分别为 2 cm 和 7 cm，圆心距 $O_1O_2 = 13$ cm，AB 是 $\odot O_1$、$\odot O_2$ 的外公切线，切点分别是 A、B.

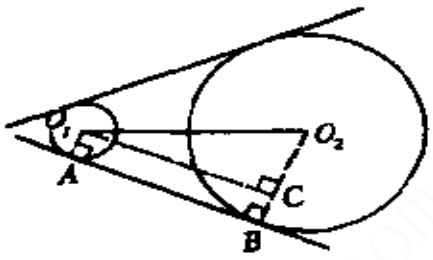


图 8.27(Figure 8.27)

求：公切线的长 AB.

解：　连结 O_1A、O_2B，则

$O_1A \perp AB$，$O_2B \perp AB$. 过 O_1 作 $O_1C \perp O_2B$，垂足为 C，则四边形 O_1ABC 为矩形，于是有

$$O_1C \perp CO_2,\ O_1C = AB,\ O_1A = CB$$

在 Rt$\triangle O_1CO_2$ 中，

$$O_1O_2 = 13,\ O_2C = O_2B - O_1A = 5$$

$$\therefore\quad O_1C = \sqrt{13^2 - 5^2} = 12\ (\text{cm})$$

$$\therefore\quad AB = 12\ \text{cm}$$

由圆的对称性知，另一条外公切线的长也为 12 cm.

Example 8.9　Given that the radii of circle O_1 and circle O_2 are 2 cm and 7 cm, respectively, central distance $O_1O_2 = 13$ cm, and AB is an exterior common tangent line of circle O_1 and circle O_2 with their tangent points A and B, respectively (see figure 8.27).

Find the length of the common tangent line AB .

Solution

Join O_1A and O_2B , then $O_1A \perp AB$, $O_2B \perp AB$. Make $O_1C \perp O_2B$ by passing through point O_1 at the perpendicular base C , and then the quadrilateral O_1ABC is a rectangle, so that

$$O_1C \perp CO_2,\ O_1C = AB,\ O_1A = CB.$$

In right angle O_1CO_2, $O_1O_2 = 13$ cm,

$O_2C = O_2B - O_1A = 5$, we obtain

$$O_1C = \sqrt{13^2 - 5^2} = 12 \text{ (cm)}.$$

Thus $AB = 12$ cm.

From the symmetry of circles, the length of the other exterior common tangent line is also 12 cm.

§8.3 圆的有关计算(Operation of Circles)

一、圆周长和弧长(Length of Circumference and Arc Length)

(一) 圆周长(Length of Circumference)

圆周长 C 与半径 R 之间有下面的关系：

$$C = 2\pi R$$

这里 $\pi = 3.14159\cdots$，这个无限不循环小数叫做**圆周率**.

The length of circumference C and radius R have the relationship as follows

$$C = 2\pi R.$$

Here, $\pi = 3.14159\cdots$, where π is a transcendental irrational number, called the **ratio of circumference** of a circle to its diameter.

(二) 弧长(Arc Length)

因为 360° 的圆心角所对的弧长就是圆周长 $C = 2\pi R$，所以 1° 的圆心角所对的弧长是 $\frac{2\pi R}{360}$，即 $\frac{\pi R}{180}$. 于是可得在半径为 R 的圆中，n° 的圆心角所对的弧长 l 的计算公式：

$$l = \frac{n\pi R}{180}$$

Since the length of an arc opposite its central angle in degree of 360° is the length of circumference with $C = 2\pi R$, the length of an arc opposite its central angle in degree of 1° is $\frac{2\pi R}{360}$, that is $\frac{\pi R}{180}$. Thus we can get, in a circle with radius R, the formula for the length of an arc l opposite its central angle in degree of $n°$ as follows

$$l = \frac{n\pi R}{180}.$$

例 8.10　如图 8.28 所示，已知圆环的外圆周长 $C_1 = 250$ cm，内圆周长 $C_2 = 150$ cm，求圆环的宽度 d（精确到 1 mm）.

解：　设外圆的半径为 R_1，内圆的半径为 R_2，

则 $d = R_1 - R_2$

$\because$ $R_1 = \frac{250}{2\pi} = \frac{125}{\pi}$

$R_2 = \frac{150}{2\pi} = \frac{75}{\pi}$

$\therefore$ $d = \frac{125}{\pi} - \frac{75}{\pi} \approx 15.9$ (cm)

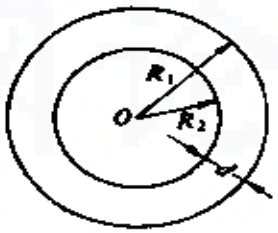


图 8.28(Figure 8.28)

Example 8.10　Given that, in figure 8.28, the length of circumference of the outside circle of the ring is $C_1 = 250$ cm and that of the inside circle is $C_2 = 150$ cm, find the width of the ring d (accurate within 1 mm).

Solution　Given that the radius of the outside circle is R_1 and that of the inside circle is R_2, then

$$d = R_1 - R_2.$$

Since

$$R_1 = \frac{250}{2\pi} = \frac{125}{\pi} \quad \text{and} \quad R_2 = \frac{150}{2\pi} = \frac{75}{\pi},$$

$$d = \frac{125}{\pi} - \frac{75}{\pi} \approx 15.9 \text{ (cm)}.$$

二、圆、扇、弓形的面积

(Areas of a Circle, Sector and Segment of Circle)

(一) 圆面积(Area of a Circle)

圆面积 S 和半径 R 之间有下面关系:

$$S = \pi R^2$$

The area of circle S and the radius R have the relationship as follows

$$S = \pi R^2.$$

(二) 扇形面积(Area of an Arc)

一条弧和经过这条弧的端点的两条半径所组成的图形叫做扇形.

A portion of a circle bounded by two radii of the circle and one of the arcs which they intersect is called a **sector** of a cirlce.

在半径为 R 的圆中,因为圆心角是 360° 的扇形面积就是圆面积 $S = \pi R^2$,所以圆心角是 1° 的扇形面积是 $\frac{\pi R^2}{360}$. 这样,在半径为 R 的圆中(图 8.29),圆心角为 n° 的扇形面积的计算公式是:

$$S_{扇形} = \frac{n}{360}\pi R^2$$

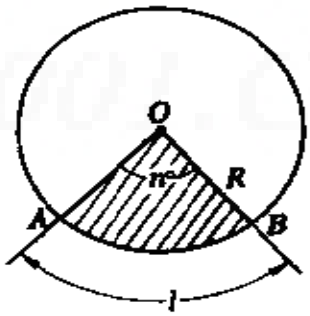


图 8.29(Figure 8.29)

又因为,扇形的弧长 $l = \frac{n\pi R}{180}$, 扇形面积 $\frac{n\pi R^2}{360}$ 可以写成 $\frac{1}{2} \cdot \frac{n\pi R}{180} \cdot R$,所以,又得到 $S_{扇形} = \frac{1}{2}lR$.

In a circle with radius R , since the area of a sector whose central

angle is 360° is the area of the circle $S = \pi R^2$, the area of a sector whose central angle is 1° is $\frac{\pi R^2}{360}$. Accordingly, in a circle with its radius R (see figure 8.29), the area of a sector whose central angle is $n°$ can be expressed in the following formula

$$S_{\text{sector}} = \frac{n}{360}\pi R^2.$$

In addition, since the length of arc of a sector $l = \frac{n\pi R}{180}$ and the area of a sector $\frac{n\pi R^2}{360}$ can be written as $\frac{1}{2} \cdot \frac{n\pi R}{180} \cdot R$, we get $S_{\text{sector}} = \frac{1}{2} lR$.

例 8.11　已知:正三角形的边长为 a,求它的内切圆与外接圆组成的圆环的面积(图 8.30).

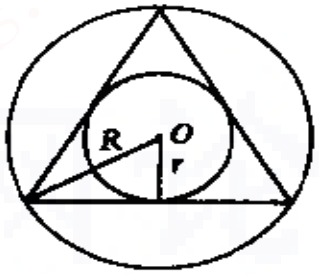


图 8.30(Figure 8.30)

解:　设正三角形外接圆、内切圆的半径分别为 R、r,面积为 S_1, S_2.

$$S = S_1 - S_2 = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

$$\because \quad R^2 - r^2 = \left(\frac{a}{2}\right)^2 = \frac{a^2}{4} \quad \therefore \quad S = \frac{\pi}{4}a^2$$

Example 8.11　Given that tha length of one side of an equilateral triangle is a, find the area of the ring consisting of an inscribed circle and a circumscribed one (see figure 8.30).

Solution　Let the radii of the circumscribed and inscribed circles be R and r, respectively, and their areas are S_1 and S_2.

$$S = S_1 - S_2 = \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

Since $R^2 - r^2 = \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$, $S = \frac{\pi}{4}a^2$.

例 8.12　如图 8.31，已知正三角形 ABC 的边长为 a. 分别与以 A、B、C 为圆心，以 $\frac{a}{2}$ 为半径的圆相切于点 O_1、O_2、O_3. 求 $\overset{\frown}{O_1O_2}$、$\overset{\frown}{O_2O_3}$、$\overset{\frown}{O_3O_1}$ 围成的图形面积 S(图中阴影部分).

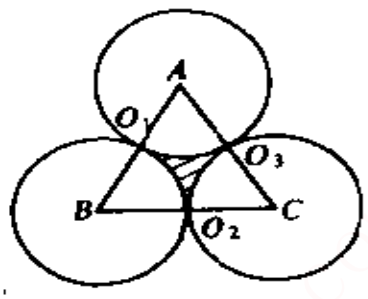


图 8.31(Figure 8.31)

解：　$S = S_{\triangle ABC} - 3S_{\text{扇形}AO_1O_3}$

$\because$　$S_{\triangle ABC} = \frac{1}{2}a \cdot \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$

$$S_{\text{扇形}AO_1O_3} = \frac{60\pi\left(\frac{a}{2}\right)^2}{360} = \frac{\pi a^2}{24}$$

$\therefore$　$S = \frac{\sqrt{3}}{4}a^2 - \frac{3\pi}{24}a^2 = \frac{2\sqrt{3}-\pi}{8}a^2$

Example 8.12　In figure 8.31, given that the length of one side of an equilateral triangle ABC is a, A, B and C are used as central points, respectively, and the circles with radii $\frac{a}{2}$ are tangent at points O_1, O_2 and O_3, find the area S of the figure surrounded by $\overset{\frown}{O_1O_2}$, $\overset{\frown}{O_2O_3}$ and $\overset{\frown}{O_3O_1}$ (the shadow of figure 8.31).

Solution　$S = S_{\triangle ABC} - 3S_{\text{sector }AO_1O_3}$.

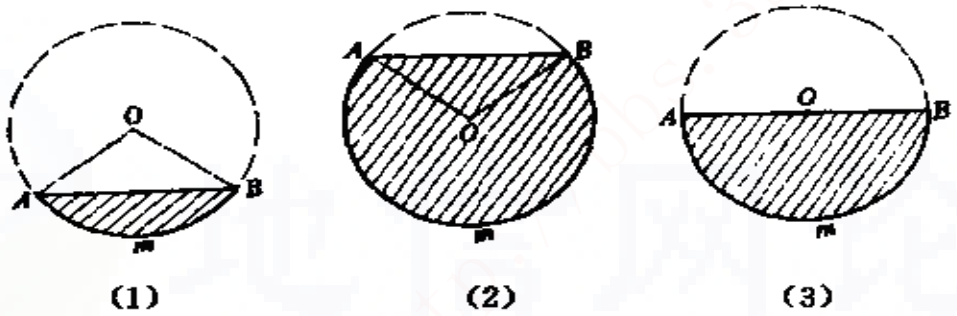
$$S_{\triangle ABC} = \frac{1}{2}a \cdot \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2,$$

$$S_{\text{sector }AO_1O_3} = \frac{60\pi\left(\frac{a}{2}\right)^2}{360} = \frac{\pi a^2}{24},$$

Therefore $S = \frac{\sqrt{3}}{4}a^2 - \frac{3\pi}{24}a^2 = \frac{2\sqrt{3}-\pi}{8}a^2$.

(三) 弓形面积(Area of Segment of a Circle)

从图 8.32 可知,把扇形 $OAmB$ 的面积以及 $\triangle OAB$ 的面积计算出来,就可以得到弓形 $\overset{\frown}{AmB}$ 的面积. 如图 8.32(1) 中,弓形 AmB 的面积小于半圆,这时 $S_{弓形} = S_{扇形} - S_{\triangle OAB}$; 图 8.32(2) 中,弓形 AmB 的面积大于半圆的面积, 这时 $S_{弓形} = S_{扇形} + S_{\triangle OAB}$; 图 8.32(3) 中, 弓形 AmB 的面积等于半圆的面积, 这时 $S_{弓形} = \frac{1}{2} S_{圆}$.



(1) (2) (3)

图 8.32(Figure 8.32)

From figure 8.32, by working out the area of sector $OAmB$ and that of equilateral triangle OAB, we can get the area of segment of a circle AmB. As shown in figure 8.32(1), the area of segment of a circle AmB is less than the halfcircle, when $S_{segment} = S_{sector} - S_{\triangle OAB}$; in figure 8.32(2), the area of segment $\overset{\frown}{AmB}$ is greater than that of its halfcircle, when $S_{segment} = S_{sector} + S_{\triangle OAB}$; in figure 8.32(3), the area of segment AmB equals the area of the halfcircle, when $S_{segment} = \frac{1}{2} S_{circle}$.

例 8.13 水平放置的圆柱形排水管截面半径是 0.6 m, 其中水面高 0.3 m. 求截面上有水的弓形面积.

解：　如图 8.33，在 ⊙O 中，连结 OA、OB，作弦 AB 的垂直平分线，垂足为 D，交$\overset{\frown}{AB}$ 于点 C.

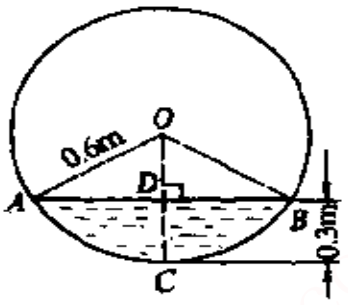


图 8.33(Figure 8.33)

∵　$OA = 0.6$，$DC = 0.3$

∴　$OD = 0.6 - 0.3 = 0.3$，$\angle AOD = 60^\circ$，$AD = 0.3\sqrt{3}$

∵　$S_{弓形ACB} = S_{扇形OACB} - S_{\triangle OAB}$

$$S_{扇形OACB} = \frac{120\pi}{360}(0.6)^2 = 0.12\pi \ (\text{m}^2)$$

$$S_{\triangle OAB} = \frac{1}{2}AB \cdot OD = \frac{1}{2} \times 0.6\sqrt{3} \times 0.3 = 0.09\sqrt{3} \ (\text{m}^2)$$

∴　$S_{弓形ACB} = 0.12\pi - 0.09\sqrt{3} \approx 0.22 \ (\text{m}^2)$.

答：截面上有水的弓形面积为 0.22 m².

Example 8.13　The radius of a section of a cylindrical drainage pipe placed horizontally is 0.6 m, of which the altitude to the water surface is 0.3 m. Find the area of the segment containing water on the section.

Solution　In circle O shown in figure 8.33, join OA and OB, make the perpendicular bisector of chord AB with the perpendicular base D and intersect $\overset{\frown}{AB}$ at point C.

Since $OA = 0.6$, $DC = 0.3$, $OD = 0.6 - 0.3 = 0.3$,

$\angle AOD = 60^\circ$, $AD = 0.3\sqrt{3}$.

Thus $S_{\text{segment } ACB} = S_{\text{sector } OACB} - S_{\triangle OAB}$,

$$S_{\text{sector } OACB} = \frac{120\pi}{360}(0.6)^2 = 0.12\pi \ (\text{m}^2),$$

$$S_{\triangle OAB} = \frac{1}{2}AB \cdot OD = \frac{1}{2} \times 0.6\sqrt{3} \times 0.3 = 0.09\sqrt{3} \ (\text{m}^2).$$

Thus $S_{\text{segment } ACB} = 0.12\pi - 0.09\sqrt{3} \approx 0.22 \ (\text{m}^2)$.

Answer The area of segment with water on the section is 0.22 m^2.

例 8.14 已知：如图 8.34，⊙O 的半径为 R，直径 $AB \perp CD$，以 B 为圆心，以 BC 为半径作$\overset{\frown}{CED}$. 求：$\overset{\frown}{CED}$ 与$\overset{\frown}{CAD}$ 围成的新月形 $ACED$ 的面积 S.

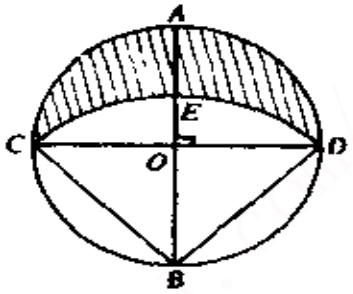


图 8.34(Figure 8.34)

解： ∵ $S = \frac{1}{2}\pi R^2 - S_{弓形CED}$

又 ∵ $S_{弓形CED} = S_{扇形BCED} - S_{\triangle BCD}$

$$S_{扇形BCED} = \frac{90\pi(BC)^2}{360} = \frac{\pi}{4}(\sqrt{2}R)^2 = \frac{\pi R^2}{2}$$

$$S_{\triangle BCD} = \frac{1}{2} \times 2R \cdot R = R^2$$

$$\therefore \quad S = \frac{\pi R^2}{2} - \frac{\pi R^2}{2} + R^2 = R^2$$

Example 8.14 Given that, in figure 8.34, the radius of circle O is R, diameter $AB \perp CD$, and make $\overset{\frown}{CED}$ with B the center and BC the radius, find the area S of the moon-shaped $ACED$ surrounded by $\overset{\frown}{CED}$ and $\overset{\frown}{CAD}$.

Solution

Since $S = \frac{1}{2}\pi R^2 - S_{\text{segment } CED}$,

and also

$$S_{\text{segment } CED} = S_{\text{sector } BCED} - S_{\triangle BCD},$$

$$S_{\text{sector } BCED} = \frac{90\pi(BC)^2}{360} = \frac{\pi}{4}(\sqrt{2}R)^2 = \frac{\pi R^2}{2}.$$

Since $S_{\triangle BCD} = \frac{1}{2} \times 2R \cdot R = R^2$,

$$S = \frac{\pi R^2}{2} - \frac{\pi R^2}{2} + R^2 = R^2.$$

第九章　空间图形的表达(Space Graphs)

§9.1　多面体(Polyhedrons)

一、棱柱(Prisms)

(一) 棱柱的概念(Concepts of Prisms)

有两个面互相平行,其余各面都是四边形,并且每相邻两个四边形的公共边都互相平行,由这些面所围成的几何图形叫做**棱柱**.如图 9.1 所示各图均为棱柱.

A prism is a polyhedron with two congruent parallel faces and whose other faces are quadrilaterals with the common sides of every two adjacent quadrilaterals parallel to each other. As is shown in figure 9.1, all the figures are prisms.

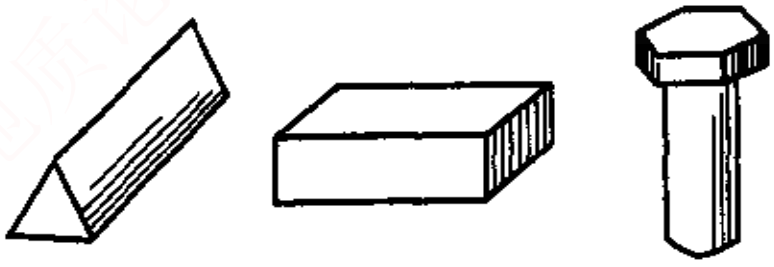


图 9.1(Figure 9.1)

如图9.2所示,两个互相平行的面叫做**棱柱的底面**,其余各面叫做**棱柱的侧面**. 两个侧面的公共边叫做**棱柱的侧棱**,侧面与底

面的公共顶点叫做**棱柱的顶点**，不在同一个面上的两个顶点的连线叫做**棱柱的对角线**，两个底面的距离叫做**棱柱的高**. 如图 9.2 中所示的棱柱，多边形 $ABCDE$ 和$A'B'C'D'E'$ 是底面，四边形 $ABB'A'$，$BCC'B'$ 等是侧面，$A'A$，$B'B$ 等是侧棱，$H'H$ 是高.

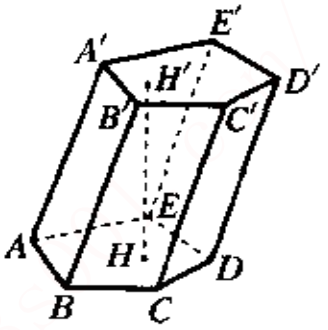


图 9.2(Figure 9.2)

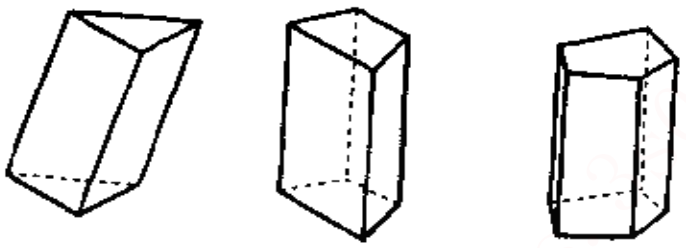
As is shown in figure 9.2, two congruent and parallel faces are called **bases of a prism** and other faces are called **lateral faces**. The intersections of lateral faces are called **lateral edges** and the common vertices joining lateral faces and bases are called **vertices**. Any line segment joining two vertices that do not lie on the same face or base is called the **diagonal of the prism** and the perpendicular distance between the two bases is called the **altitude**. As is shown in figure 9.2, in the prism, polyhedrons $ABCDE$ and $A'B'C'D'E'$ are the bases, quadrilaterals $ABB'A'$, $BCC'B'$, etc., are lateral faces, $A'A$, $B'B$, etc., are lateral edges, $H'H$ is the altitude.

棱柱用表示底面各顶点的字母来表示，如图 9.2 所示的棱柱记作棱柱 $ABCDE - A'B'C'D'E'$，或用表示一条对角线端点的两个字母来表示，例如棱柱 AC'.

A prism is expressed in letters representing each vertex of bases. For instance, the prism in figure 9.2 is written as prism $ABCDE - A'B'C'D'E'$, or in two letters representing the endpoints of a diagonal, such as prism AC'.

侧棱不垂直于底面的棱柱叫做**斜棱柱**(如图 9.3(1))，侧棱垂直于底面的棱柱叫做**直棱柱**(如图 9.3(2))；底面是正多边形的直棱柱叫做**正棱柱**(如图 9.3(3)).

The prism whose lateral edges are not perpendicular to the bases is called a **slant prism** (see figure 9.3(1)) and the prism whose lateral edges are perpendicular to the bases is called a **right prism** (see figure 9.3(2)). The right prism with its base a regular polygon is called a **regular prism** (see figure 9.3(3)).



(1) (2) (3)

图 9.3(Figure 9.3)

棱柱的底面可以是三角形,四边形,五边形,……我们把这样的棱柱分别叫做**三棱柱**(图 9.3(1))、**四棱柱**(图 9.3(2))、**五棱柱**(图 9.3(3)).

The bases of a prism can be a triangle, a quadrilateral, a pentagon, etc., and we call these prisms a **triangular prism** (figure 9.3(1)), a **quadrangular prism** (figure 9.3(2)) and a **pentagonal prism** (figure 9.3(3)).

棱柱的性质(Properties of a Prism)

(1) 侧棱都相等,侧面是平行四边形.

The lateral edges are equal and the lateral faces are parallel quadrilaterals.

(2) 两个底面与平行于底面的截面是全等的多边形.

Two bases and the intersections parallel to the bases are congruent quadrilaterals.

(3) 过不相邻的两条侧棱的截面是平行四边形.

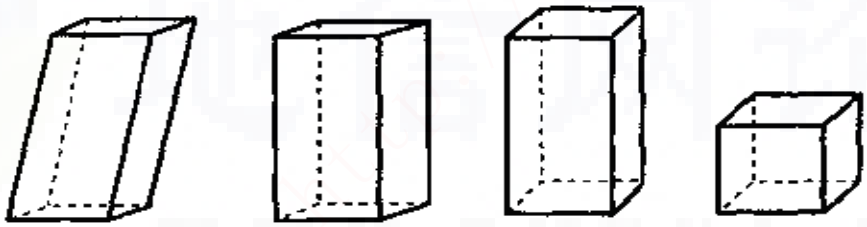
The intersections passing through two lateral edges which are not adjacent are parallel quadrilaterals.

(二) 长方体{Rectangular Solids}

长方体是四棱柱的特殊情形.

A rectangular solid is a special case of a quadrangular prism.

底面是平行四边形的四棱柱叫做**平行六面体**(图 9.4(1)). 侧棱与底面垂直的平行六面体叫做**直平行六面体**(图 9.4(2)). 底面是矩形的直平行六面体叫做**长方体**(图 9.4(3)). 棱长都相等的长方体叫做**正方体**(图 9.4(4)).



(1) (2) (3) (4)

图 9.4(Figure 9.4)

The quadrangular prism with bases parallel quadrilaterals is called a **parallel hexahedron** (figure 9.4(1)). The parallel hexahedron with lateral edges perpendicular to bases is called a **right parallel hexahedron** (figure 9.4(2)). The right parallel hexahedron whose bases are rectangles is called a **rectangular solid** (figure 9.4(3)). The rectangular solid whose lateral edges are all equal is called a **tetragonal solid** (figure 9.4(4)).

定理(Theorem)

长方体一条对角线长的平方等于一个顶点上三条棱长的平方和.

The square of length of a diagonal in a rectangular solid equals the sum of square of the lengths of three lateral edges at a vertex.

例 9.1　已知长方体 AC', $B'D$ 是一条对角线(图 9.5).

求证:

$B'D^2 = AB^2 + BC^2 + BB'^2$

证明:连结 BD.

$\because$ $B'B \perp BD$

$\therefore$ $B'D^2 = BD^2 + BB'^2$

又 $\because$ $BD^2 = AB^2 + AD^2 = AB^2 + BC^2$

$\therefore$ $B'D^2 = AB^2 + BC^2 + BB'^2$

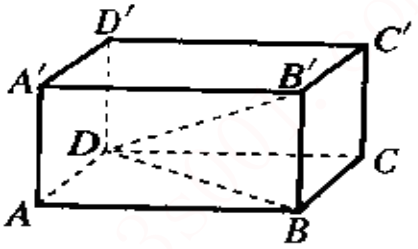


图 9.5(Figure 9.5)

Example 9.1　Given that, in rectangular solid AC', $B'D$ is a diagonal (figure 9.5), illustrate $B'D^2 = AB^2 + BC^2 + BB'^2$.

Illustration　Join BD.

Since $B'B \perp BD$, $B'D^2 = BD^2 + BB'^2$.

Also since $BD^2 = AB^2 + AD^2 = AB^2 + BC^2$,

$B'D^2 = AB^2 + BC^2 + BB'^2$.

(三) 直棱柱的侧面积(Lateral Area of a Right Prism)

直棱柱的侧面展开是矩形(图 9.6),这个矩形的长等于直棱柱底面周长 C,宽等于直棱柱的高 h,由此可以得到如下定理:

如果直棱柱的底面周长是 C,高是 h,那么它的侧面积是

$$S_{\text{直棱柱侧}} = Ch$$

棱柱的全面积等于侧面积与两底面积的和.

The expanded graph of lateral sides of a right prism is a rectangle whose length equals the perimeter C of the base of the right prism (see figure 9.6), and whose width equals the altitude h of the right prism, from which we can get the following theorem:

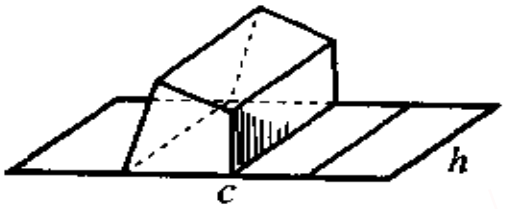


图 9.6(Figure 9.6)

If the perimeter of the base of a right prism is C and the altitude h, then its lateral area is

$$S_{\text{lateral area}} = Ch.$$

The whole area of a prism equals the sum of lateral area and two base area.

二、棱锥(Pyramids)

(一) 棱锥的概念(Concepts of Pyramids)

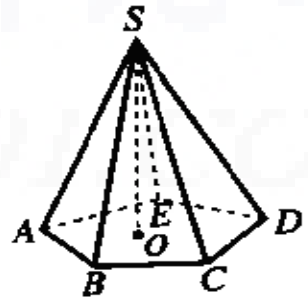


图 9.7(Figure 9.7)

有一个面是多边形,其余各面是有一个公共顶点的三角形,由这些面所组成的图形叫做**棱锥**(如图 9.7).这个多边形叫做棱锥的**底面**,其余各面叫做棱锥的**侧面**,相邻侧面的公共边叫做棱锥的**侧棱**,各侧面的公共顶点叫做棱锥的**顶点**,顶点到底面的距离叫做棱锥的**高**.如图 9.7 中所示的棱锥,多边形 $ABCDE$ 是底面,三角形 SAB, SBC 等是侧面,SA, SB 等是侧棱,S 是顶点,SO 是高.

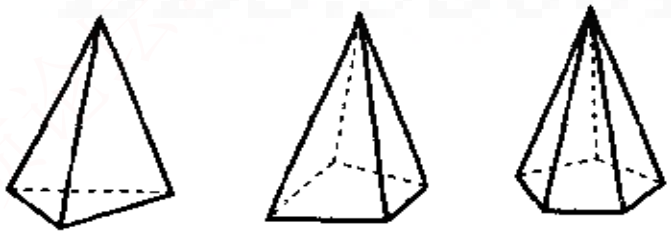
A polyhedron with one face a polygon and the other faces triangles containing a common vertex is called a **pyramid** (see figure 9.7). The polygon is called the **base** of the pyramid, the triangles are called the

lateral faces, the common sides of the adjacent lateral faces are called **lateral edges**, the common vertex of each lateral face is called the **vertex** and the distance from the vertex to the base is called the **altitude** of the pyramid. For instance, in figure 9.7, polygon *ABCDE* is the base, triangles *SAB*, *SBC*, etc. are lateral faces, *SA*, *SB*, etc. are lateral edges, *S* is the vertex and *SO* the altitude.

棱锥用表示顶点和底面各顶点，或者底面一条对角线端点的字母来表示. 例如，对于图 9.7 中所示的棱锥，可以记作棱锥 *S-ABCDE*，或者棱锥 *S-AC*.

A pyramid is expressed in the letters representing its common vertex and each vertex on the base or in the letters representing the endpoints of a diagonal on the base, such as the pyramid in figure 9.7, which can be written as pyramid *S-ABCDE* or pyramid *S-AC*.

棱锥的底面可以是三角形，四边形，五边形……. 因此我们把这样的棱锥分别叫做三棱锥(图 9.8(1))、四棱锥(图 9.8(2))、五棱锥(图 9.8(3))、…….



(1) (2) (3)

图 9.8(Figure 9.8)

The base of a pyramid can be a triangle, quadrilateral, pentagon, etc. So we call these pyramids a triple pyramid (figure 9.8(1)), a quadruple

pyramid (figure 9.8(2)), a quintuple pyramid (figure 9.8(3)), etc.

(二) 正棱锥(Regular Pyramids)

如果一个棱锥的底面是正多边形,并且顶点在底面的射影是底面中心,这样的棱锥叫做**正棱锥**.

The pyramid with a regular polygon for base and with the foot of its altitude at the center of the base is called a **regular pyramid**.

正棱锥有下面一些性质:

(A regular pyramid has the following properties)

1. 各侧棱相等,各侧面都是全等的等腰三角形,各等腰三角形底边上的高相等,它叫做正棱锥的**斜高**.

Each of the lateral edges is equal and all lateral faces are congruent isosceles triangles with equal altitude on the base side which is called the **slant height** of a regular pyramid.

2. 棱锥的高、斜高和斜高在底边上的射影组成一个直角三角形;棱锥的高、侧棱和侧棱在底面上的射影也组成一个直角三角形(图 9.9).

The altitude of a pyramid, its slant height and the project on the base side make a right triangle; the altitude of a pyramid, its lateral edges and the project on the base side make a right triangle, too (see figure 9.9).

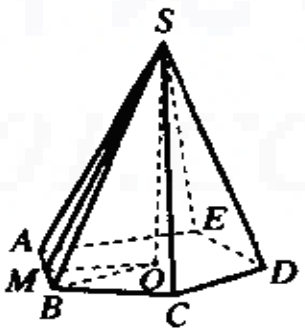


图 9.9(Figure 9.9)

3. 如果棱锥被平行于底面的平面所截,那么截面和底面相似,并且它们的面积的比等于截得的棱锥的高和已知棱锥的高的平方比.该性质也适用于一般棱锥.

If a pyramid is sectioned by a plane parallel to the base, the section is similar to the base and the ratio of their areas equals the ratio of square of the altitude of the sectioned pyramid to the square of the altitude of the known pyramid. This property is also applied to general pyramids.

例 9.2　如图 9.10,已知正三棱锥 S-ABC 的高 $SO = h$,斜高 $SM = l$. 求经过 SO 的中点平行于底面的截面 $\triangle A'B'C'$ 的面积.

解：　连结 OM、OA,在 Rt$\triangle SOM$ 中,$OM = \sqrt{l^2 - h^2}$. 因为棱锥 $S - ABC$ 是正棱锥,所以点 O 是正三角形 ABC 的中心.

$$AB = 2AM = 2 \cdot OM \cdot \tan 60^\circ$$
$$= 2\sqrt{3} \cdot \sqrt{l^2 - h^2}$$
$$S_{\triangle ABC} = \frac{\sqrt{3}}{4} AB^2$$
$$= \frac{\sqrt{3}}{4} \times 4 \times 3(l^2 - h^2)$$
$$= 3\sqrt{3}(l^2 - h^2)$$

根据一般棱锥截面的性质,有

$$\frac{S_{\triangle A'B'C'}}{S_{\triangle ABC}} = \frac{h'^2}{h^2} = \frac{1}{4}$$

$$\therefore \quad S_{\triangle A'B'C'} = \frac{3\sqrt{3}}{4}(l^2 - h^2)$$

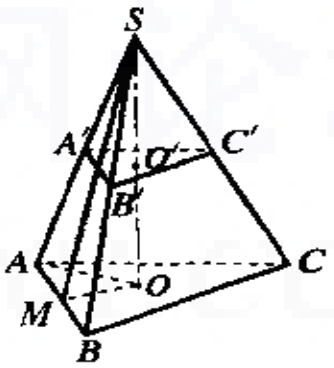


图 9.10(Figure 9.10)

Example 9.2　In figure 9.10, given that the altitude $SO = h$ of a regular triangular pyramid S-ABC and slant height $SM = l$, get the area of section $\triangle A'B'C'$ passing through the central point of line SO and parallel to the base.

Solution　Join OM and OA, and in right triangle SOM, $OM = \sqrt{l^2 - h^2}$. Since pyramid $S - ABC$ is a regular

pyramid, point O is the center of regular triangle ABC.

$$AB = 2AM = 2 \cdot OM \cdot \tan 60^\circ$$
$$= 2\sqrt{3} \cdot \sqrt{l^2 - h^2},$$
$$S_{\triangle ABC} = \frac{\sqrt{3}}{4} AB^2 = \frac{\sqrt{3}}{4} \times 4 \times 3(l^2 - h^2)$$
$$= 3\sqrt{3}(l^2 - h^2).$$

In terms of the property of sections of general pyramids, have

$$\frac{S_{\triangle A'B'C'}}{S_{\triangle ABC}} = \frac{h'^2}{h^2} = \frac{1}{4}.$$

Therefore $S_{\triangle A'B'C'} = \frac{3\sqrt{3}}{4}(l^2 - h^2)$.

(三) 正棱锥的侧面积(Lateral Area of a Regular Pyramid)

棱锥的侧面展开图是由各个侧面组成的,展开图的面积就是棱锥的侧面积.正 n 棱锥的侧面展开图如图9.11所示.设正 n 棱锥的底面边长为 a,周长为 c,斜高为 h',则展开图的面积为 $n \cdot \frac{1}{2} ah' = \frac{1}{2} ch'$. 由此得到下面的定理:

如正棱锥的底面周长是 c,斜高是 h',那么它的侧面积是

$$S_{\text{正棱锥侧}} = \frac{1}{2} ch'$$

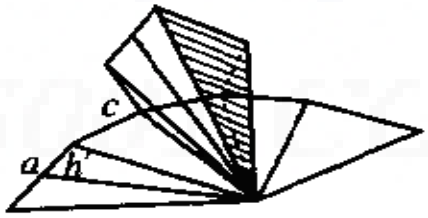


图 9.11(Figure 9.11)

棱锥的全面积等于侧面积与底面积的和.

The lateral expanded graph of a pyramid is composed of every lateral side and the area of the graph is just the combination of the lateral area of the pyramid. The lateral expanded graph of a n -side regular pyramid is seen in figure 9.11. Given that the length of the base side is a, the perimeter c and the slant height h', then the area of the expanded graph

is $n \cdot \frac{1}{2}ah' = \frac{1}{2}ch'$, from which we get the following theorem:

If the perimeter of the base of a regular pyramid is c and its slant height h', then its lateral area is

$$S_{\text{lateral area}} = \frac{1}{2}ch'.$$

The whole area of a pyramid equals the sum of its lateral area and base area.

例 9.3　设计一个正四棱锥形冷水塔塔顶，高是 0.85 m，底的边长是 1.5 m，制造这种塔顶需要多少米铁板？

解：　如图 9.12，S 表示塔顶的顶点，O 表示底的中心，则 SO 是高．设 SE 是斜高．

在直角三角形 SOE 中，根据勾股定理得

$$SE = \sqrt{\left(\frac{1.5}{2}\right)^2 + 0.85^2}$$
$$\approx 1.13\ (\text{m})$$

$$\therefore \quad S_{\text{正棱锥侧}} = \frac{1}{2}ch'$$
$$= \frac{1}{2}(1.5 \times 4) \times 1.13$$
$$\approx 3.4\ (\text{m}^2)$$

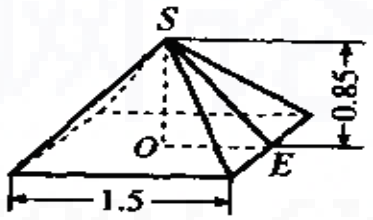


图 9.12(Figure 9.12).

答：　制造这种塔顶需要铁板约 3.4 m^2.

Example 9.3　Design a roof of a cooling tower in a regular rectangular pyramid, whose altitude is 0.85 m and length of the base side 1.5 m. How many square meters does this iron plate need for making such a roof of tower?

Solution　In figure 9.12, S is the vertex of the tower top, O is the center of the base and SO is the altitude. let SE be the slant height.

In right triangle SOE, in terms of the Pythagorean theorem

$$SE = \sqrt{\left(\frac{1.5}{2}\right)^2 + 0.85^2} \approx 1.13\ (\text{m}).$$

$$\therefore \quad S_{\text{lateral area}} = \frac{1}{2}ch' = \frac{1}{2}(1.5\times 4)\times 1.13 \approx 3.4\ (\text{m}^2).$$

Answer The iron plate needed for this tower top is about 3.4 m^2.

三、棱台(The Frustum of Pyramid)

(一) 棱台的概念(Concepts of Frustum of Pyramid)

用一个平行于棱锥底面的平面去截棱锥,底面和截面之间的部分叫做**棱台**,原棱锥的底面和截面叫做棱台的**下底面**和**上底面**,其它各面叫做棱台的**侧面**. 相邻侧面的公共边叫做棱台的**侧棱**. 上、下底面之间的距离叫做棱台的**高**. 如图 9.13 中的棱台,多边形 $A'B'C'D'$ 和 $ABCD$ 是上、下底面,四边形 $ABB'A'$、$BCC'B'$ 等是侧面,$A'A$、$B'B$ 等是侧棱,$O'O$ 是它的高.

Using a plane parallel to the base of a pyramid to cut the pyramid, we get a section of a pyramid between the base and the plane which is called a **frustum of pyramid**. The base and the intersection of the original pyramid are called the **bottom base** and the **upper base** of the frustum of the pyramid and other faces are called **lateral faces**. The common side of adjacent lateral faces is called a **lateral edge** of the frustum of the pyramid. The distance between the upper and bottom bases is called the **altitude**. Such that, in the frustum shown in figure 9.13, polygons $A'B'C'D'$ and $ABCD$ are upper and bottom bases, quadrilaterals $ABB'A'$, $BCC'B'$, etc. lateral faces, $A'A$, $B'B$, etc. lateral edges and $O'O$ the altitude.

棱台用表示上、下底面各顶点的字母来表示,如图 9.13 所示的棱台,可记作棱台 $ABCD - A'B'C'D$. 或者用它的对角线端点字

母表示，如棱台 AC'.

A frustum can be expressed in the letters representing each of the vertices on the upper and bottom bases such as the frustum in figure 9.13 which can be written as frustum $ABCD - A'B'C'D$ or in the letters of the endpoints on the diagonal, such as the frustum AC'.

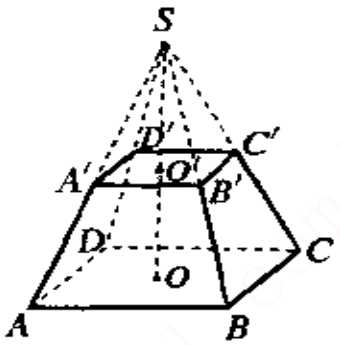


图 9.13(Figure 9.13)

由三棱锥、四棱锥、五棱锥……截得的棱台，分别叫做**三棱台、四棱台、五棱台**……

Frustums sectioned by a triangle pyramid, a rectangular pyramid, a pentagon pyramid, etc. are called **triangle frustum, rectangular frustum, pentagon frustum**, etc., respectively.

(二) 正棱台(Regular Frustums)

由正棱锥截得的棱台叫做**正棱台**.

The frustum sectioned by a regular pyramid called a **regular frustum**.

正棱台的性质(Properties of Regular Frustums)

1. 正棱台的侧棱相等，侧面是全等的等腰梯形，各等腰梯形的高相等，它叫做正棱台的**斜高**.

The lateral egdes of a regular frustum are equal, lateral faces are congruent isosceles trapezoids, the altitudes, which are called the **slant height** of the regular frustum, of each isosceles trapezoid are equal.

2. 正棱台的两底面以及平行于底面的截面是相似正多边形.

The two bases of a regular frustum and the intersection parallel to the

bases are similar regular polygons.

3. 正棱台的两底面中心连线、相应的边心距和斜高组成一个直角梯形；两底面中心连线、侧棱和两底面相应的半径也组成一个直角梯形(图 9.14).

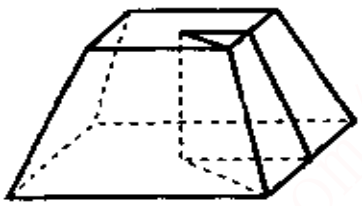


图 9.14(Figure 9.14)

The central link line between two bases of a regular frustum, its corresponding apothem and slant height form a right angled trapezoid; the central link line between two bases, lateral egdes and appropriate radii of the two bases also form a right-angled trapezoid (figure 9.14).

(三) 正棱台的侧面积(Lateral Area of a Regular Frustum)

棱台的侧面展开图是由各个侧面组成的. 展开图的面积就是棱台的侧面积，如图 9.15 所示. 设它的上、下底面边长是 a'、a，边数是 n，斜高是 h'，那么它的侧面积是 $n \cdot \frac{1}{2}(a + a')h' = \frac{1}{2}(na + na')h'$，由于 na'、na 分别是上、下底面的周长 c'、c，我们得到下面的定理：

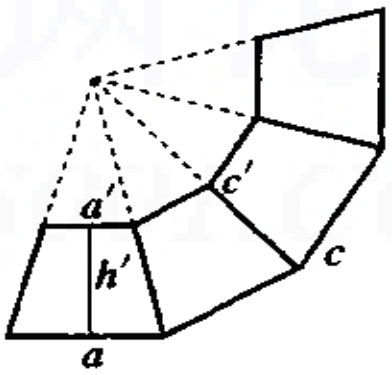


图 9.15(Figure 9.15)

如果正棱台的上、下底面的周长是 c'、c，斜高是 h'，那么它的侧面积是

$$S_{\text{正棱台侧}} = \frac{1}{2}(c + c')h'$$

The lateral expanded graph of a frustum is composed of each of the

lateral faces. The area of the graph is just lateral area of the frustum. As is shown in figure 9.15, provided that the lengths of the sides of the upper and bottom bases are a' and a, the number of sides is n and the slant height h', then the lateral area is $n \cdot \frac{1}{2}(a + a')h' = \frac{1}{2}(na + na')h'$ and since na' and na are the perimeters c' and c of the upper and bottom bases, respectively, we get the theorem as follows:

If the perimeters of the upper and bottom bases of a regular frustum are c' and c, and the slant height is h', then its lateral area is

$$S_{\text{lateral area}} = \frac{1}{2}(c + c')h'.$$

棱台的全面积等于它的侧面积与上、下底面积的和.

The whole area of a frustum is equal to the sum of its lateral area and the areas of its upper and bottom bases.

例 9.4　粉碎机上的下料斗是正四棱台形(图 9.16),它的两底面边长分别是 80 mm 和 440 mm,高是 200 mm. 计算制造这样一个下料斗所需铁板的面积.

解: 上底面周长

$$c' = 4 \times 80 = 320\ (\text{mm})$$

下底面周长

$$c = 4 \times 440 = 1,760\ (\text{mm})$$

斜高 $h' = \sqrt{200^2 + \left(\frac{440 - 80}{2}\right)^2}$

$$\approx 269\ (\text{mm})$$

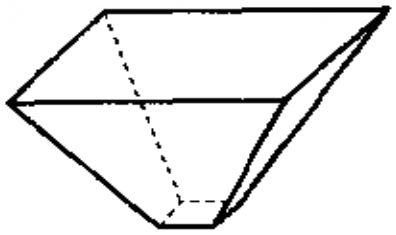


图 9.16(Figure 9.16)

$$\therefore \quad S_{\text{正棱台侧}} = \frac{1}{2}(c + c')h'$$

$$= \frac{1}{2}(320 + 1,760) \times 269 \approx 2.8 \times 10^5(\text{mm}^2)$$

答：制造这样一个下料斗需铁板约 2.8×10^5 mm^2.

Example 9.4 The blanking bin of a breaker is a regular rectangular frustum (figure 9.16), the side lengths of whose two bases are 80 mm and 440 mm, respectively, and whose altitude is 200 mm. Calculate the area of the iron plate needed for manufacturing such a blanking bin.

Solution The perimeter of upper base $c' = 4 \times 80 = 320$ (mm)

The perimeter of bottom base $c = 4 \times 440 = 1,760$ (mm)

Slant height $h' = \sqrt{200^2 + \left(\frac{440 - 80}{2}\right)^2} \approx 269$ (mm)

Therefore, $S_{\text{lateral area}} = \frac{1}{2}(c + c')h'$

$= \frac{1}{2}(320 + 1,760) \times 269 \approx 2.8 \times 10^5$ (mm^2).

Answer The iron plate needed for making such a blanking bin is about 2.8×10^5 mm^2.

在正棱台的侧面积公式中，如果设 $c' = c$，就得到正棱柱的侧面积公式：$S_{\text{正棱柱侧}} = ch'$（这里 h' 是高）；如果设 $c' = 0$，就得到正棱锥的侧面积公式：$S_{\text{正棱锥侧}} = \frac{1}{2}ch'$.

In the formula of the lateral area of a regular frustum, letting $c' = c$, we obtain the formula of the lateral area of a regular prism $S_{\text{lateral faces}} = ch'$ (here h' is the altitude); letting $c' = 0$, we obtain the formula of the area of a regular pyramid $S_{\text{lateral area}} = \frac{1}{2}ch'$.

四、多面体(Polyhedrons)

由若干个多边形所围成的几何体叫做**多面体**. 棱柱、棱台、棱

锥等都是多面体.围成多面体的各个多边形叫做多面体的**面**,两个面的公共边叫做多面体的**棱**,若干个面的公共顶点叫做多面体的**顶点**. 许多矿物结晶体都是多面体形的.(图 9.17)

The geometric solid bounded by some plane polygons is called a **polyhedron**. The prism, frustum and pyramid are all polygons. Each of the polygons forming the polyhedron is called the **face** of the polyhedron; the common side of two faces is called the **edge** of the polyhedron, and the common vertices where three or more edges intersect are called the **vertices** of the polyhedron. Many mineral crystals are the form of polyhedrons.(see figure 9.17)



图 9.17(Figure 9.17)

把多面体的任何一个面伸展为平面,如果所有其他各面都在这个平面的同侧,这样的多面体叫做**凸多面体**.所有的棱柱、**棱锥**、棱台指的都是凸多面体.

The polyhedron which lies all on one side of any plane containing one of its faces is called a **convex polyhedron**. All the prism, pyramid and frustum are convex polyhedrons.

一个多面体至少有四个面,多面体依照它的面数分别叫做**四面体**、**五面体**、**六面体**等.例如,三棱锥是四面体,三棱柱是五面体,正方体是六面体等.

A polyhedron has at least four faces and, in terms of the number of its faces, it can be called a **tetrahedron**, **pentahedron**, **hexahedron**,

etc. For instance, a triangular pyramid is a tetrahedron, a triangular prism a penrahedron and a regular cube a cube hexahedron.

§9.2 旋转体(The Solid of Rotation)

一、圆柱、圆锥、圆台
(Circular Cylinders, Cones and Truncated Cones)

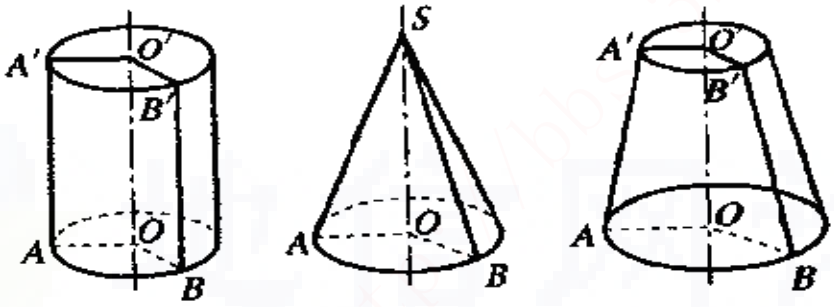


图 9.18(Figure 9.18)

(一) 定义(Definitions)

分别以一矩形的一边、直角三角形一直角边、直角梯形垂直于底边的腰所在的直线为**旋转轴**,其余各边旋转而形成的**曲面**所围成的几何体分别叫做**圆柱、圆锥、圆台**(图 9.18). 旋转轴叫做它们的**轴**,在轴上这条边的长度叫做它们的**高**,垂直于轴的边旋转而成的**圆**面叫做它们的**底面**,不垂直于轴的边旋转而成的**曲面**叫做它们的**侧面**,无论旋转到什么位置,这条边都叫做**侧面的母线**,如图 9.18 中,直线 $O'O$、SO 是高,$A'A$、$B'B$、SA、SB 等是母线.

The geometric solid generated by revolving a plane about a line (the **axis of revolution**) is called a solid of rotation. The solid of rotation revolving about a side of a rectangle is called a **circular cylinder**, that

revolving about a leg of a right triangle is called a **circular cone** and that revolving about the side perpendicular to the base of a right-angled trapezoid is called a **circular-truncated cone** (figure 9.18). The axes of revolution are called their **axes**; the lengths of sides on their axes are called their **altitudes**; the circular surfaces by revolving about the side perpendicular to the axes are called their **bases** and the curved surfaces by revolving about the side which is not perpendicular to the axes are called their **lateral surfaces**. No matter what position it revolves, the side is called the **generatrix of lateral surface**, such that, in figure 9.18, line $O'O$ and SO are altitudes, and $A'A$, $B'B$, SA, SB, etc. are generatrices.

(二) 圆柱、圆锥、圆台的性质(Properties of Circular Cylinders, Cones and Truncated Cones)

1. 平行于底面的截面都是圆.

Sections parallel to the bases are all circles.

2. 过轴截面(轴截面)分别是全等的矩形、等腰三角形、等腰梯形.

The sections passing through the axes (axis sections) are congruent rectangles, isosceles triangles and isosceles trapezoids, respectively.

例 9.5 把一个圆锥截成圆台,圆台上、下底面半径的比是 1:4,母线长是 10 cm,求圆锥的母线长.

解: 设圆锥的母线长为 y,圆台上、下底面半径分别是 x、$4x$(图 9.19),根据相似三角形的比例关系,得

$$(y - 10) : y = x : 4x$$

也就是 $4(y - 10) = y$, $3y = 40$

$\therefore \quad y = \frac{40}{3}$ (cm)

因此，圆锥的母线长为$\frac{40}{3}$ cm.

Example 9.5 Section a circular cone into a circular truncated cone. Given that the ratio of the radius on the upper base to that on the bottom base of the circular truncated cone is 1 : 4 and the length of its generatrix is 10 cm, find the length of the generatrix of the circular cone.

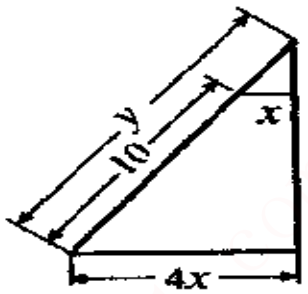


图 9.19(Figure 9.19)

Solution Let the length of the generatrix of the circular cone be y, the radii of both the upper and bottom bases are x and $4x$, respectively (see figure 9.19), and in terms of the proportion of similar triangles, get

$$(y - 10) : y = x : 4x.$$

That is $4(y - 10) = y, \quad 3y = 40.$

Thus $y = \frac{40}{3}$ (cm).

As a result, the length of the generatrix of the circular cone is $\frac{40}{3}$cm.

(三) 圆柱、圆锥、圆台的侧面积(Lateral Areas of Circular Cylinders, Cones and Truncated Cones)

把圆柱、圆锥、圆台的侧面沿着它们的一条母线剪开后展开在平面上，展开图的面积就是它们的侧面积.

Extend the lateral surfaces of a circular cylinder, cone and truncated cone on a plane along one of their generatrices and the areas of the expanded graphs are their lateral areas.

1. 圆柱的侧面积(Lateral Area of a Cylinder)

图 9.20 是圆柱的侧面展开图,它是一个矩形,这个矩形的长等于圆柱底面周长 c,宽等于圆柱面的母线长 l(也就是高),由此可得定理:

如果圆柱底面半径是 r,周长是 c,侧面母线长是 l,那么它的侧面积是

$$S_{\text{圆柱侧}} = cl = 2\pi rl$$

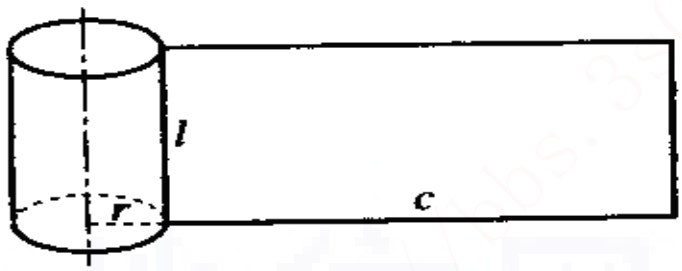


图 9.20(Figure 9.20)

Figure 9.20 is an expanded graph of the lateral surface of a circular cylinder, which is a rectangle whose length is equal to the perimeter c of the base of the cylinder and whose width is equal to the length l (that is the altitude) of the generatrix of the surface of the cylinder, so we get the theorem as follows:

If the radius of the base of a circular cylinder is r, the perimeter c and the length of lateral generatrix l, then its lateral area is

$$S_{\text{lateral area}} = cl = 2\pi rl.$$

2. 圆锥侧面积(Lateral Area of a Circular Cone)

图 9.21 是圆锥的侧面展开图,它是一个扇形,这个扇形的弧长等于圆锥底面的周长 c,半径等于圆锥侧面的母线长 l,由此可得定理:

如果圆锥底面半径是 r,周长是 c,侧面母线长为 l,那么它的侧面积是

$$S_{\text{圆锥侧}} = \frac{1}{2}cl = \pi rl$$

Figure 9.21 is an expanded graph of the lateral surface of a circular cone which is a sector whose arc length equals the perimeter c of the base of the circular cone and whose radius the length l of the generatrix of its lateral surface, from which we get the theorem as follows:

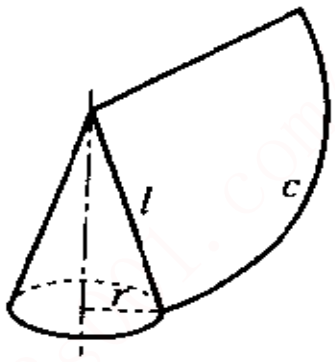


图 9.21(Figure 9.21)

If the radius of the base of a circular cone is r, the perimeter c and the length of the lateral generatrix l, then its lateral area is

$$S_{\text{lateral area}} = \frac{1}{2}cl = \pi rl.$$

3. 圆台的侧面积(Lateral Area of a Circular Truncated Cone)

图 9.22 是圆台的侧面展开图,通常把这样的图形叫做**扇环**,由扇环可以求出圆台的侧面积.

Figure 9.22 is an expanded graph of the lateral surface of a circular truncated cone which is usually called **a sector ring** from which we can get the lateral area of the truncated cone.

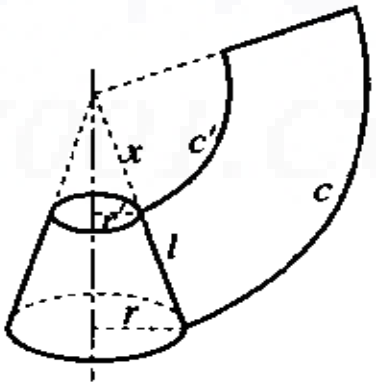


图 9.22(Figure 9.22)

设圆台侧面母线长为 l,上、下底面周长分别为 c'、c,半径分别是 r'、r,于是

$$S_{\text{圆台侧}} = \frac{1}{2}c(l+x) - \frac{1}{2}c'x = \frac{1}{2}[cl + (c-c')x] \quad (1)$$

$\because \quad \frac{c'}{c} = \frac{x}{x+l} \quad \therefore \quad x = \frac{c'l}{c-c'}$

代入(1),得

$$S_{\text{圆台面}} = \frac{1}{2}\left[cl + (c-c')\frac{c'l}{c-c'}\right]$$

$$= \frac{1}{2}(c+c')l = \pi(r+r')l$$

由此我们得到下面的定理:

如果圆台的上、下底面半径是 r'、r,周长是 c'、c,侧面母线长是 l,那么它的侧面积是

$$S_{\text{圆台侧}} = \frac{1}{2}(c+c')l = \pi(r+r')l$$

Let the length of the lateral generatrix of a circular truncated cone be l, the perimeters of the upper and bottom bases are c' and c and their radii are r' and r, respectively, then

$$S_{\text{lateral area}} = \frac{1}{2}c(l+x) - \frac{1}{2}c'x = \frac{1}{2}[cl + (c-c')x]. \quad (1)$$

Since $\frac{c'}{c} = \frac{x}{x+l}$, $x = \frac{c'l}{c-c'}$.

Bring it to (1) and get

$$S_{\text{lateral area}} = \frac{1}{2}\left[cl + (c-c')\frac{c'l}{c-c'}\right]$$

$$= \frac{1}{2}(c+c')l = \pi(r+r')l.$$

Consequently, we get the following theorem:

If the radii of the upper and bottom bases of a truncated cone are r' and r, the perimeters are c' and c, respectively, and the length of the generatrix of its lateral surface is l, then its lateral area is

$$S_{\text{lateral area}} = \frac{1}{2}(c+c')l = \pi(r+r')l.$$

圆柱、圆锥、圆台的全面积分别等于它们的侧面积与底面积的和.

The whole area of a circular cylinder, cone and truncated cone is equal to the sums of their lateral areas and base areas, respectively.

例 9.6　圆锥的底面半径为 r，侧面母线长为 l，侧面展开图扇形的圆心角为 θ°.

求证：$\theta = \frac{r}{l} \cdot 360$

证明：　图 9.23 是圆锥侧面展开图，因为扇形的弧长等于圆锥底面的周长，即 $\frac{\pi l\theta}{180} = 2\pi r$.

所以　$\theta = \frac{r}{l} \cdot 360$

Example 9.6　The radius of the base of a circular cone is r, the length of the generatrix of its lateral surface is l and the central angle of the sector for the expanded graph of the lateral surface is θ°. Illustrate $\theta = \frac{r}{l} \cdot 360$.

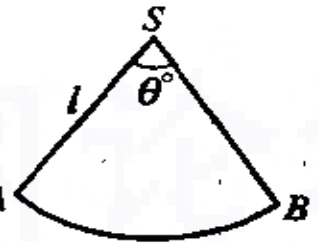


图 9.23(Figure 9.23)

Illustration　Figure 9.23 is an expanded graph of the lateral surface of the circular cone. Since the arc length of the sector equals the perimeter of the base of the cone, that is

$$\frac{\pi l\theta}{180} = 2\pi r.$$

Thus　$\theta = \frac{r}{l} \cdot 360.$

在圆台的侧面积公式中，如果设 $c' = c$，就得到圆柱侧面积

公式 $S_{圆柱侧} = cl$. 如果设 $c' = 0$, 就得到圆锥侧面积公式 $S_{圆锥侧} = \frac{1}{2}cl$.

In the formula of lateral area of a circular truncated cone, if $c' = c$, we get the formula of lateral area for a circular cylinder $S_{\text{lateral area}} = cl$. If $c' = 0$, we get the formula of lateral area for a circular truncated cone $S_{\text{lateral area}} = \frac{1}{2}cl$.

二、球(Spheres)

(一) 球的概念(Concepts of Spheres)

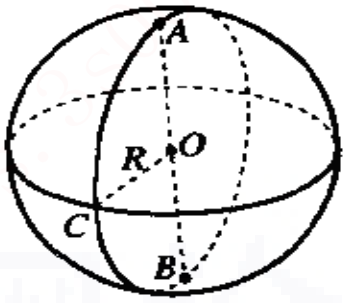


图 9.24(Figure 9.24)

半圆以它的直径为旋转轴, 旋转所成的曲面叫做**球面**. 球面所围成的几何体叫做**球体**, 简称**球**. 半圆的圆心叫做**球心**. 连结球心和球面上任意一点的线段叫做球的**半径**. 连结球面上两点并且经过球心的线段叫做**球的直径**. 如图 9.24 的球中, 点 O 是球心, 线段 OC 是球的半径, 线段 AB 是球的直径.

The curved surface formed with a half-circle revolving about its diameter which is used as an axis of revolution is called a **sphere**. The geometric solid surrounded by the sphere is called a **spheroid**, simplified as a sphere. The center of the half-circle is the **center of spheroid**. The segment joining the center and any point on the spherical surface is called the **radius of spheroid**. The segment joining two points on the spherical surface through the center is called the **diameter of sphere**. As is shown in figure 9.24, point O is the center, segment OC the radius and AB the diameter of the sphere.

球面也可以看作与定点(球心) 的距离等于定长(半径) 的所

有点的集合(轨迹).一个球用表示它的球心的字母来表示,例如球 O.

A sphere can also he looked upon as the locus of points in space at a given distance from a fixed point. It is expressed in the letter to represent its center, such as sphere O.

(二) 球截面的性质(Properties of Spherical Sections)

用一个平面去截一个球,截面是圆面.球的截面有下面的性质:

1. 球心和截面圆心的连线垂直于截面(图 9.25);

2. 球心到截面的距离 d 与球的半径 R 及截面的半径 r 有下面的关系:

$$r = \sqrt{R^2 - d^2}$$

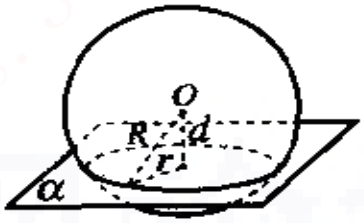


图 9.25(Figure 9.25)

Using a plane to cut a sphere, the resulting section is a circular surface. The section of a sphere has the following properties:

1. The joint line hetween the center of a sphere and that of the section is perpendicular to the section (figure 9.25);

2. There is the following relationship hetween the distance d from the center of the sphere to the section and the radius R of the sphere, and the radius r of the section as well:

$$r = \sqrt{R^2 - d^2}.$$

球面被经过球心的平面截得的圆叫做**大圆**,被不经过球心的截面截得的圆叫做小圆.

The circle intersected by a plane passing through the center of a sphere is called a **great circle** and that not passing through the center is

called a **small circle**.

在球面上，两点之间的最短距离，就是经过这两点的大圆在这两点间的一段劣弧的长度．这个弧长叫做**两点的球面距面**．如图 9.26 中，$\overset{\frown}{PQ}$ 的长度就是 P、Q 两点的球面距离．

On the spherical surface, the shortest distance between two points is just the length of an inferior arc between these two points through the great circle, which is called the **distance between two points**. For example, in figure 9.26, the length of $\overset{\frown}{PQ}$ is the distance between points P and Q of the spherical surface.

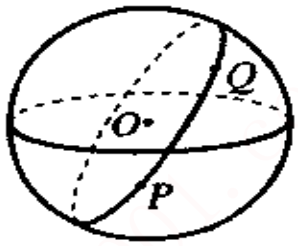


图 9.26(Figure 9.26)

(三) 球的表面积(Surface Area of a Sphere)

球面面积等于它的大圆面积的 4 倍．

$$S_{\text{球面}} = 4\pi R^2$$

The area of a spherical surface is equal to four times the area of a great circle of a sphere.

$$S_{\text{area of a sphere}} = 4\pi R^2.$$

例 9.7　已知：圆柱的底面直径与高都等于球的直径(图 9.27)．求证：(1) 球的表面积等于圆柱的侧面积；(2) 球的表面积等于圆柱全面积的 $\frac{2}{3}$．

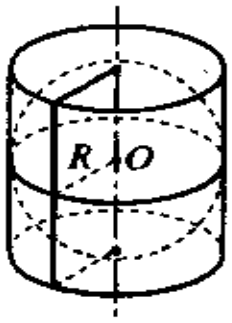


图 9.27(Figure 9.27)

证明：

(1) 设球的半径为 R，则圆柱的底面半径为 R，高为 $2R$，得

$$S_{球} = 4\pi R^2$$

$$S_{圆柱侧} = 2\pi R \cdot 2R = 4\pi R^2$$

$$\therefore \quad S_{球} = S_{圆柱侧}$$

(2) $\because$ $S_{圆柱全} = 4\pi R^2 + 2\pi R^2 = 6\pi R^2$

$$S_{球} = 4\pi R^2$$

$$\therefore \quad S_{球} = \frac{2}{3} S_{圆柱全}$$

Example 9.7 Given, in figure 9.27, that the base diameter and the altitude of a circular cylinder are both equal to the diameter of a ball, illustrate (1) the surface area of the ball equals the lateral area of the circular cylinder; (2) the surface area of the ball equals two-thirds of the whole area of the circular cylinder.

Illustration

(1) Let the radius of the ball be R, then the base radius of the circular cylinder is R and the altitude is $2R$, getting

$$S_{\text{ball}} = 4\pi R^2, \quad S_{\text{cylinder}} = 2\pi R \cdot 2R = 4\pi R^2,$$

Therefore $S_{\text{ball}} = S_{\text{cylinder}}$.

(2) Since $S_{\text{whole cylinder}} = 4\pi R^2 + 2\pi R^2 = 6\pi R^2$, $S_{\text{ball}} = 4\pi R^2$,

$$S_{\text{ball}} = \frac{2}{3} S_{\text{whole cylinder}}.$$

三、球冠(Spherical Crowns)



图 9.28(Figure 9.28)

(一) 球冠(Spherical Crowns)

球面被平面所截得的一部分叫做**球冠**,截得的圆叫做**球冠的底**,垂直于截面的直径被截得的一段叫做**球冠的高**,如图 9.28.

The segment of a sphere intersected by a plane is called a **spherical crown**, the intersected circle is called the **base of spherical crown** and the segment of the diameter perpendicular to the section is called the **altitude of crown**, such as the crown in figure 9.28.

球冠也可以看作是一段圆弧绕经过它的一个端点的直径旋转所成的曲面.

A spherical crown can also be regarded as the curved surface with a circular arc revolving about the diameter passing through one of its endpoints.

设球半径为 R,球冠的高为 h,则球冠的面积等于截成它的球面上大圆周长与球冠的高的积.

$$S_{球冠} = 2\pi Rh$$

Let the radius of a sphere be R and the altitude of a spherical crown h, then the area of the spherical crown is equal to the perimeter of the great circle times the altitude of the spherical crown.

$$S_{\text{spherical crown}} = 2\pi Rh.$$

这个公式,对于小于半球面的球冠、半球面、大于半球面的球冠及整个球面都是适用的.当 $h = 2R$ 时,就是球面的面积 $4\pi R^2$.

This formula is suitable for any of the spherical crown less than a half-sphere, the spherical crown greater than a half-sphere, as well as the whole sphere. When $h = 2R$, the area is just the area of a sphere, that is $4\pi R^2$.

例 9.8　运油车的油罐是由一个圆筒与两个相同的球冠部分组成的.油罐的尺寸如图 9.29(单位 m).求制造这样一个油罐需要多少平方米钢板.

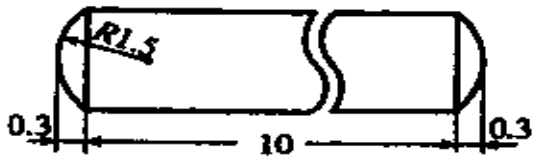


图 9.29(Figure 9.29)

解： 设圆筒半径为 r，它也是球冠的底的半径. 由平面几何知识可得

$$r^2 = 0.3(2 \times 1.5 - 0.3)$$

$$\therefore \quad r = 0.9\ (\text{m})$$

油罐圆筒部分面积：

$$S_1 = 2\pi r h_1 = 2\pi \times 0.9 \times 10 \approx 56.52\ (\text{m}^2)$$

两个球冠部分的面积：

$$2S_2 = 2 \times 2\pi R h_2 = 2 \times 2\pi \times 1.5 \times 0.3 \approx 5.65\ (\text{m}^2)$$

油罐的总面积：

$$S = S_1 + 2S_2 = 56.52 + 5.65 \approx 62.2\ (\text{m}^2)$$

答： 制造这样一个油罐要用钢板约 62.2 m^2.

Example 9.8 A tank of an oil truck consists of a cylinder and two spherical crown-shaped containers. The dimension of the tank is shown in figure 9.29 (unit is meter). Find how much steel plate is needed for manufacturing such a tank.

Solution Let the radius of the cylinder be r, which is also the radius of the base of the spherical crown. In terms of the principles of plane geometry, we can get

$$r^2 = 0.3(2 \times 1.5 - 0.3).$$

Thus $r = 0.9\ (\text{m})$.

The area of the cylinder:

$$S_1 = 2\pi r h_1 = 2\pi \times 0.9 \times 10 \approx 56.52\ (\text{m}^2)$$

The area of two crown-shaped containers:

$$2S_2 = 2 \times 2\pi Rh_2 = 2 \times 2\pi \times 1.5 \times 0.3 \approx 5.65\ (\text{m}^2).$$

The whole area of the tank:

$$S = S_1 + 2S_2 = 56.52 + 5.65 \approx 62.2\ (\text{m}^2).$$

Answer The steel plate for manufacturing such an oil tank is about 62.2 m^2.

(二) 旋转面和旋转体 (Surface and Solid of Rotation)

一条平面曲线(包括直线)绕它所在的平面内的一条定直线旋转所形成的曲面叫做**旋转面**. 这条定直线叫做**旋转轴**. 无论旋转到什么位置,这条曲线都叫做旋转面的**母线**. 图 9.30 中,直线 a 是**旋转轴**,曲线 l(不论旋转到什么位置)是母线.

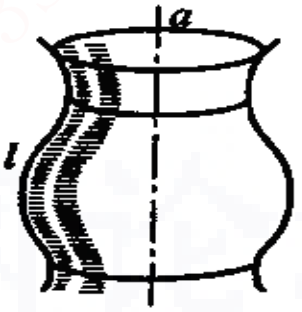
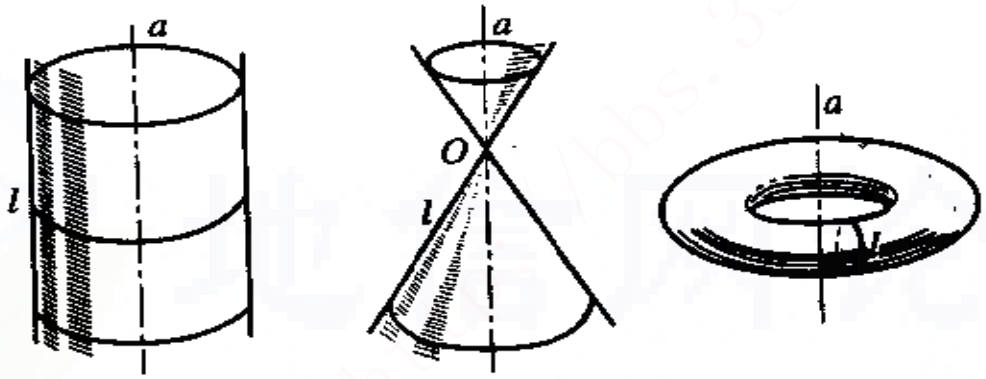


图 9.30(Figure 9.30)

The curved surface with one plane curve (including a straight line) revolving about a fixed line in a plane is called the **surface of rotation**. This fixed line is called the **axis of rotation**. No matter where the axis goes, this resulting curve is always called a **generatrix** of the surface of rotation. In figure 9.30, line a is an axis of rotation and curve l (no matter where it is) is a generatrix.

如果母线是与旋转轴平行的直线,那么形成的旋转面叫做**圆柱面**(图 9.31(1)). 如果母线是和旋转轴斜交的直线,那么形成的旋转面叫做**圆锥面**(图 9.31(2)),这时,母线和轴的交点叫做圆锥面的**面点**. 如果一个圆绕同一平面与它不相交的一条直线旋转,那么形成的旋转面叫做**环面**(图 9.31(3)).

If the generatrix is a line parallel to the axis of rotation, then the resulting surface of rotation is called the **surface of cylinder** (see figure 9.31(1)). If the generatrix is a line slantingly intersecting the axis of rotation, then the resulting surface of rotation is called the **surface of circular cone** (figure 9.31(2)), when the intersection point of a generatrix and an axis is called the **vertex** of the surface of the circular cone. If a circle revolves around a line which does not intersect it about the same plane, then the resulting surface of rotation is called a **ring surface** (figure 9.31(3)).



(1) (2) (3)

图 9.31(Figure 9.31)

封闭的旋转面围成的几何体,叫做**旋转体**.这时,旋转面的轴也叫做**旋转体的轴**.圆柱、圆锥、圆台、球都是旋转体.环面所围成的几何体也是旋转体,它叫做**环体**,简称**环**.

The geometric solid surrounded by a closed surface of rotation is called the **solid of rotation**. In this case, the axis of the surface of rotation is also called the **axis of solid of rotation**. Circular cylinders, circular cones, circular truncated cones and spheres are all solids of rotation. The geometric solid surrounded by the ring surface is also the solid of rotation which is called the **ring solid**, simplified as the **ring**.

§9.3 多面体和旋转体的体积

(The Volume of Polyhedrons and Solids of Rotation)

一、体积的概念(The Concepts of Volume)

几何体占有空间部分的大小叫做它的体积.

The volume refers to the size in which a geometric solid takes up the space.

要度量一个几何体的体积,首先要选取一个单位体积作为标准,然后求出几何体的体积是单位体积的多少倍,这个倍数就是这个几何体的体积数值. 通常取棱长等于单位长度(如 1 cm,1 m 等)的正方体的体积作为体积单位(图 9.32).

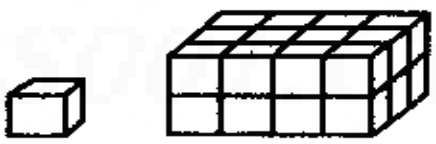


图 9.32(Figure 9.32)

To measure the volume of a geometric solid, firstly, choose a unit volume as a standard, then find how many times the volume of the solid is taken by unit volume, and the times is the quantity of the volume of this geometric solid. Usually choose as the unit volume the volume of a tetragonal solid with the length of edge equal to the unit length, such as 1 cm and 1 m.(figure 9.32).

设长方体的长、宽、高分别为 a、b、c,长方体的体积等于它的长、宽、高的积.

$$V_{长方体} = abc$$

Let the length, width and altitude of a rectangular solid be a, b and c, respectively, and then the volume of the rectangular solid equals the product of its length, width and altitude.

$$V_{\text{rectangular solid}} = abc.$$

设长方体的底面积为 S，高为 h，则长方体的体积等于 S 与 h 的积，即

$$V_{长方体} = Sh$$

Let the base area of a rectangular solid be S and altitude h, then the volume of the rectangular solid is equal to the product of S and h, that is

$$V_{\text{rectangular solid}} = Sh.$$

正方体的体积等于它的棱长 a 的立方，即

$$V_{长方体} = a^3$$

The volume of a rectangular solid is equal to the cube of a, the length of the edge, that is

$$V_{\text{rectangular solid}} = a^3.$$

二、棱柱、圆柱的体积

(The Volume of Prisms and Circular Cylinders)

柱体(棱柱、圆柱)的体积等于它的底面积 S 与高 h 的乘积.

$$V_{柱体} = Sh$$

The volume of a cylindrical body (a prism, a circular cylinder) is equal to the product of its base area S and altitude h.

$$V_{\text{cylinder}} = Sh.$$

底面半径是 r，高为 h 的圆柱的体积是

$$V_{圆柱} = \pi r^2 h$$

The volume of a circular cylinder with the base radius r and altitude h is

$$V_{\text{circular cylinder}} = \pi r^2 h.$$

例 9.9　有一堆相同规格的六角螺帽毛坯(图 9.33)共重 5.8 kg. 已知底面六边形的边长是 12 mm，高是 10 mm，内孔直径是 10 mm. 问约有毛坯多少个(铁的密度是 7.8 g/cm^3)?

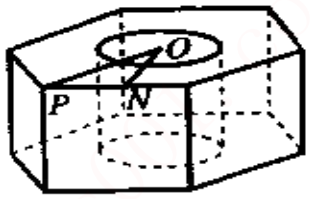


图 9.33(Figure 9.33)

解：　六角螺帽毛坯的体积是一个正六棱柱的体积与一个圆柱的体积的差.

$$V_{正六棱柱} = \frac{\sqrt{3}}{4} \times 12^2 \times 6 \times 10 \approx 3.74 \times 10^3 (\text{mm}^3)$$

$$V_{圆柱} = 3.14 \times \left(\frac{10}{2}\right)^2 \times 10 \approx 0.785 \times 10^3 (\text{mm}^3)$$

毛坯的体积

$$V = 3.74 \times 10^3 - 0.785 \times 10^3$$
$$\approx 2.96 \times 10^3 (\text{cm}^3) = 2.96\ (\text{cm}^3)$$
$$5.8 \times 10^3 \div (7.8 \times 2.96)$$
$$\approx 2.5 \times 10^2 (\text{个})$$

答：　这堆毛坯约有 250 个.

Example 9.9　A pile of hex nut blanks (figure 9.33) weigh 5.8 kg. Given that the length of the sides on the base of a hexagon is 12 mm, the altitude 10 mm and the internal diameter 10 mm, find how many pieces of the blank there

are (the density of iron is 7.8 g/cm^3).

Solution The volume of a hex nut blank is the difference of the volume of a regular hexagonal prism and that of a circular cylinder.

$$V_{\text{regular hexagonal prism}} \approx \frac{\sqrt{3}}{4} \times 12^2 \times 6 \times 10 \approx 3.74 \times 10^3 (\text{mm}^3),$$

$$V_{\text{circular cylinder}} = 3.14 \times \left(\frac{10}{2}\right)^2 \times 10 \approx 0.785 \times 10^3 (\text{mm}^3).$$

The volume of a blank

$$V = 3.74 \times 10^3 - 0.785 \times 10^3$$
$$\approx 2.96 \times 10^3 (\text{mm}^3) = 2.96\ (\text{cm}^3).$$

$$5.8 \times 10^3 \div (7.8 \times 2.96) \approx 2.5 \times 10^2 (\text{个}).$$

Answer There are about 250 blanks in the pile.

三、棱锥、圆锥的体积
(The Volume of Pyramids and Circular Cones)

如果一个锥体(棱锥、圆锥)的底面积是 S,高是 h(图 9.34),那么它的体积是

$$V_{\text{锥体}} = \frac{1}{3}Sh$$

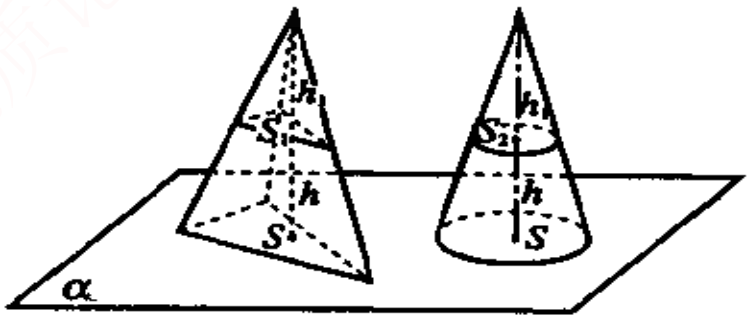


图 9.34(Figure 9.34)

If the base area of a cone (pyramid, circular cone) is S and the altitude h (figure 9.34), then its volume is

$$V_{\text{cone}} = \frac{1}{3}Sh.$$

如果圆锥的底面半径是 r,高是 h,那么它的体积是

$$V_{\text{圆锥}} = \frac{1}{3}\pi r^2 h$$

If the radius of the base of a circular cone is r and the altitude h, then its volume is

$$V_{\text{circular cone}} = \frac{1}{3}\pi r^2 h.$$

例 9.10　一块正方形薄铁板的边长是 22 cm,以它的一个顶点为圆心,边长为半径画弧,沿弧剪下一个扇形,用这块扇形铁板围成一个圆锥筒,求它的容积.

解：　如图 9.35,扇形弧长是 $\frac{1}{4}\cdot 44\pi = 11\pi$. 因此,所作的圆锥筒底的周长 $2\pi r = 11\pi$. 解得 $r = 5.5$.

因为母线长是 22 cm,所以圆锥的高

$$h = \sqrt{22^2 - 5.5^2}$$
$$\approx 21.3$$
$$V_{\text{圆锥}} = \frac{1}{3}\pi \times 5.5^2 \times 21.3$$
$$\approx 6.7 \times 10^2 (\text{cm}^3)$$

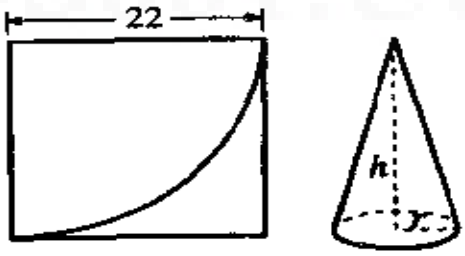


图 9.35(Figure 9.35)

答:所求圆锥筒的容积约为 670 cm^3.

Example 9.10　The length of one side of square-shaped iron plate is 22 cm, draw an arc by using a vertex of the square as the center and a side as the radius and cut a sector along

the arc, and then use this sector of the plate to make a cone cylinder. Find its volume.

Solution In figure 9.35, the arc length of the sector is $\frac{1}{4} \cdot 44\pi = 11\pi$. Therefore, the perimeter of base of the cylinder $2\pi r = 11\pi$. Solve for $r = 5.5$.

Since the length of the generatrix is 22 cm, the altitude of the circular cone is

$$h = \sqrt{22^2 - 5.5^2} \approx 21.3.$$

$$V_{\text{circular cone}} = \frac{1}{3}\pi \times 5.5^2 \times 21.3 \approx 6.7 \times 10^2 (\text{cm}^3).$$

Answer The volume of the circular cone is about 670 cm^3.

四、棱台、圆台的体积(The Volume of Frustums of a Pyramid and Cone)

如果台体(棱台、圆台)的上、下底面的面积分别是 S'、S,高是 h(图 9.36),那么它的体积是

$$V_{台体} = \frac{1}{3}h(S + \sqrt{SS'} + S')$$

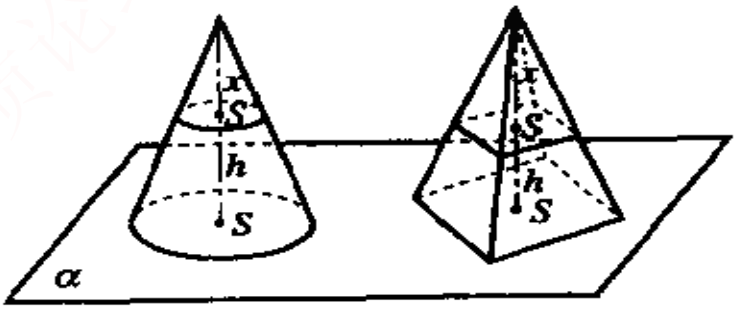


图 9.36(Figure 9.36)

If the areas of the upper and lower bases of a frustum (for a pyramid or a cone) are S' and S , respectively (see figure 9.36), and the altitude is h , then its volume is

$$V_{\text{frustum}} = \frac{1}{3}h(S + \sqrt{SS'} + S').$$

如果圆台的上、下底面半径分别是 r'、r,高是 h,那么它的体积是

$$V_{\text{圆台}} = \frac{1}{3}\pi h(r^2 + rr' + r'^2)$$

If the radii of the upper and lower bases of a truncated cone are r' and r, respectively, and the altitude is h, then its volume is

$$V_{\text{truncated cone}} = \frac{1}{3}\pi h(r^2 + rr' + r'^2).$$

在台体的体积公式中,如果设 $S' = S$, 就得到柱体的体积公式 $V_{\text{柱体}} = Sh$; 如果设 $S' = 0$,就得到锥体的体积公式 $V_{\text{锥体}} = \frac{1}{3}Sh$.

In the formula of the volume of a frustum, if $S' = S$, we get the formula of the volume of a cylindrical body $V_{\text{cylinder}} = Sh$; if $S' = 0$, we get the formula of the volume of a truncated cone $V_{\text{cone}} = \frac{1}{3}Sh$.

例 9.11　有一个正四棱台形油槽,可以装煤油 190 L,假如它的两底面边长分别等于 60 cm 和 40 cm,求它的深度.

解：　上底面面积 $S' = 40^2 = 1,600$

下底面面积 $S = 60^2 = 3,600$

$$\sqrt{S \cdot S'} = \sqrt{40^2 \times 60^2} = 2,400$$

$$\therefore \quad V = \frac{1}{3}h(3,600 + 2,400 + 1,600) = \frac{7,600}{3}h$$

由已知　$V = 190\ \text{L} = 190{,}000\ \text{cm}^3$

$\therefore \quad h = \dfrac{3 \times 190{,}000}{7{,}600} = 75\ (\text{cm})$

答：　油槽深度是 75 cm.

Example 9.11　There is an oil bath of a regular rectangular frustum, which can contain 190 L of kerosene. Assuming the side lengths of both bases are equal to 60 cm and 40 cm, respectively, find its depth.

Solution　The area of the upper base $S' = 40^2 = 1{,}600$;

the area of its lower base $S = 60^2 = 3{,}600$,

$$\sqrt{S \cdot S'} = \sqrt{40^2 \times 60^2} = 2{,}400,$$

Thus $V = \dfrac{1}{3}h(3{,}600 + 2{,}400 + 1{,}600) = \dfrac{7{,}600}{3}h$.

From the known　$V = 190\ \text{L} = 190{,}000\ \text{cm}^3$,

get $h = \dfrac{3 \times 190{,}000}{7{,}600} = 75\ (\text{cm})$.

Answer　The depth of the oil bath is 75 cm.

五、球的体积(The Volume of Spheres)

如果球的半径是 R,那么它的体积是

$$V_{球} = \frac{4}{3}\pi R^3$$

If the radius of a sphere is R, then its volume is

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3.$$

例 9.12　有一种空心钢球,重 142 g,测得外径等于 5.0 cm. 求它的内径(钢密度是 7.9 g/cm^3).

解：　设空心钢球的内径为 $2x$,那么钢球的质量是

$$7.9 \cdot \left[\frac{4}{3}\pi \cdot \left(\frac{5}{2}\right)^3 - \frac{4}{3}\pi x^3\right] = 142$$

$$x^3 = \left(\frac{5}{2}\right)^3 - \frac{142 \times 3}{7.9 \times 4\pi} \approx 11.3$$

$\therefore \quad x \approx 2.24, \quad 2x \approx 4.5$ (cm)

答：空心钢球的内径约为 4.5 cm.

Example 9.12 There is a hollow steel ball which weighs 142 g and whose external diameter is 5.0 cm. Find its internal diameter (the density of steel is 7.9 g/cm^3).

Solution Let the internal diameter of the ball be $2x$, then its mass

$$7.9 \cdot \left[\frac{4}{3}\pi \cdot \left(\frac{5}{2}\right)^3 - \frac{4}{3}\pi x^3\right] = 142,$$

$$x^3 = \left(\frac{5}{2}\right)^3 - \frac{142 \times 3}{7.9 \times 4\pi} \approx 11.3.$$

Thus $x \approx 2.24, \quad 2x \approx 4.5$ (cm).

Answer The internal diameter of the hollow steel ball is about 4.5 cm.

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